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Exploring the Universal Extra Dimensions

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Abstract

The idea of living in more than four spacetime dimensions stems from the quest for unifying electromagnetism with gravity. Later, it has been realized that such a scenario could account for the so-called hierarchy problem as well as the existence of a particle such as a dark matter candidate, none of which has yet been addressed by the Standard Model of particle physics. In this project, we will explore the Standard Model within a five-dimensional (5D) spacetime framework by considering an additional compact space dimension. After briefly reviewing the available alternatives, we concentrate on the minimal version of the so-called universal extra dimensions. Taking the gauge invariant and renormalizable Lagrangian in 5D, we first obtain its 4D effective form, which brings about a tower of the so-called Kaluza-Klein states for each Standard Model particle. Then the Feynman rules of the model are partly extracted in the general R_ξ gauge and are compared with the ones available in the literature. Due to time constrictions, this project has merely served as an introductory tool for a more profound work which we are willing to study in the future, and thus the effects of the Kaluza-Klein states on the Higgs and top quark physics will be investigated accordingly.

Contents

1 Introduction

On an evolutionary basis, in the presence of a gravitational field which simply makes objects fall down near the surface of the Earth, we have been driven to come to the conclusion that we live in a three (space) dimensional world. Gravity breaks the symmetry between the vertical and the horizontal, and in order to differentiate left from right, we most probably used the fact that the nature was not isotropically symmetric due to the existence of daily physical objects or due even to the Sun rising from one direction and lowering in the opposite. We may not have experienced a supernatural event that could alter our view of the world in terms of the number of spacetime dimensions. However, the picture was to be modified at the beginning of the 20th century, just before the World War 1.

The idea of living in more than three (space) dimensions is apparently 102 years old now. It has its roots from a genuine attempt of unifying electromagnetism and gravity into a single theory. The first to give it a try was Gunnar Nordstrom, a Finnish theoretical physicist, in 1914 [1]. Nordstrom failed in his attempts since he had a scalar gravitational potential in mind. Starting with a presumably earlier version of a general-relativity-like theory, Nordstrom first promoted the Poisson equation to the four spacetime dimensions as early as 1912 [2], and then finally displayed a genuine attempt at unifying electromagnetism and gravity two years later. Not clearly at his time, his works were bound to be doomed by the theory of general relativity (GR) of Albert Einstein in 1916 [3]. When physicists realized that Einstein's tensor theory of gravity had more to offer without razing the Nordstrom theory to the ground, Theodor Kaluza [4] and Oskar Klein [5, 6] extended GR to a five-dimensional (5D) framework, with the extra dimension constrained on a circle of some small radius. Kaluza made use of Hermann Weyl's ideas. In 1918, Weyl discovered a new metric, besides the usual one $g_{\mu\nu}$ and, to his understanding, it was the electromagnetic potential A_μ, and finally he showed that both of the known forces at the time – electromagnetism and gravity – have a common origin. Although Kaluza took a path which requires a working knowledge of GR, Klein presented somewhat better comprehensible approach. He compactified the extra dimension (ED) on a circle of a small radius, the bigger picture of which is directly applicable to the action of the Standard Model (SM) fields. The collaborative works of Kaluza and Klein are referred to as Kaluza-Klein (KK) theories.

Later, physicists studying the original KK theories faced with some problems [7]: Firstly, as stated above, a *scalar* field representing the gravity seems to couple to matter. Secondly, only if radius of the circle of compactification, R, is of the order of the fundamental scale of quantum gravity, M_* ($R \sim M_*^{-1}$), is the gauge field strength is of order one. Thirdly and finally, when the standard model is promoted to five dimensions, fermions lose their chirality, which yields a doubling of fermions for the massless case. In order to overcome these problems and produce a fruitful attempt at a unification, string theories thrived and dominated the world of extra dimensions [1]. In 1960s, Bryce Seligman DeWitt managed to reduce 4D Yang-Mills theories from more than five dimensions, and did a similar work regarding strings in 26 dimensions. In 1970s and 1980s, it was realized that the superstring theories necessitate the existence of 10 dimensions. The maximum number of dimensions in supergravity was determined to be 11. In 1990s, the number of dimensions for superstrings within the framework of the M theory were promoted to 11.

An era of renaissance started in 1998 for the theory of ED's. The models which were built in this era are claimed to solve the hierarchy problem[1] in the SM. The first one is referred to as the

[1]Let us explain the hierarchy problem by giving two examples. The fundamental Planck scale is enormously larger than the electroweak scale. The Planck scale can be written in terms of the Newton constant G_N, and the electroweak scale in terms of the Fermi constant, G_F. We observe that $G_N/G_F \sim 10^{-34}$. So far, we cannot experimentally account for this *mismatching* scales [1]. Another example is the large differences in fermion masses,

ADD model, after the authors Nima Arkani-Hamed, Savas Dimopoulos, and Gia Dvali [9, 10]. In this model, the authors assert that the fundamental scale of quantum gravity can be lowered to the TeV scale if the SM fields are localized to a real world ($(3+1)$D) surface (a *brane*) in a world with higher dimensions. The main trait of this model is that the ED's are compactified and this compactification is not with a small radius or ED in general. That is, the ED's occupy a large (though compact) volume, and effectively the strength of gravity will be reduced from the fundamental to Planck scale [7]. We like to imagine this worldview as in that there is a *sheet* hanging from a laundry line – representing the real $(3+1)$D world – and there a chunk of ED's all compactified, forming a ball-shape object attached to the sheet. The size of the *ball* and the floating-motion of the *sheet* are details irrelevant at the moment. What is most relevant to our topic is that matter, Higgs, and gauge fields – all the SM fields – live in the sheet (the brane), but only the gravity is allowed to survive in all dimensions (the *bulk*). The bulk spacetime is assumed to be flat. This model is also referred to as the model of "large extra dimensions." ADD proposal is important since it promises to solve some old problems, such as the problem of cosmological constant [1]. As a side note, we observe that the ADD model makes sense only with greater than or equal to two ED's, provided that the fundamental scale of quantum gravity is taken to be 1 TeV. To illustrate, the size of the ED's are about 1 mm with two ED's, and it goes down to 10^{-11} m with six ED's [7].

The year after, Lisa Randall and Raman Sundrum published their ED model [11]. The Randall-Sundrum (RS) model consists of two branes, just like two sheets hanging from two different laundry lines facing each other. One brane has dimensionful parameters which are scaled to TeV scale. This is where the SM is confined to live. The other brane is at the Planck scale. Moreover, the branes are connected. This connection is provided by a *warped* ED. The background metric has a warp factor in the form of $e^{-k|y|}$ (so the bulk spacetime is not flat), where y is the fifth coordinate that is not specified on a circle this time. Instead, the ED coordinate is defined on an interval. The Z_2 symmetry is again imposed here, that is, the model-builders make the identification $y \to -y$. Similarly to the ADD model, it is only the gravitational field that lives in the ED's.

In 2001, there appeared a model named the universal extra dimensions (UED) in a paper by Thomas Appelquist, Hsin-Chia Cheng, and Bogdan A. Dobrescu [12]. In the model, the very first thing to note is that all the SM fields are allowed to live in all the dimensions (hence the adjective *universal*). The spacetime of the ED's is flat, and the ED's are, again, compactified[2]. In UED model, the ED's have radius $1/R \sim 300\,\mathrm{GeV}$ with one ED, and $1/R \sim 500\,\mathrm{GeV}$ with two ED's[3]. The UED model has more to offer than only proposing a reformulation of the hierarchy problem in the SM; the fifth component of the 5-gradient imposes a conservation law – the conservation of the fifth component of 5-momentum. It turns out that the conservation of 5-momentum is heavily related to the conservation of KK numbers. It was realized that in the interaction of

which is then called the fermion mass hierarchy [8].

[2]In theories with odd number of spacetime dimensions, there appears a chirality problem of fermions. One way to overcome this issue is to do an orbifolding, that is, compactify the extra dimensions. In the ADD and UED models alike, there is a compactification on the circle with a Z_2 symmetry. This seems to be working so far [7].

[3]The physicist in the field of high energy physics tend to express most, if not all, the measurements in units of energy. Considering the beginners, we may need to *feel* how large, for example, some distance $d = (1\,\mathrm{GeV})^{-1}$ is. As for the length conversion in natural units, we have the equivalence [13]

$$(1\,\mathrm{GeV})^{-1} \equiv 0.197 \times 10^{-15}\,\mathrm{m}$$

and hence we may simply take, for approximation purposes,

$$(1\,\mathrm{GeV})^{-1} \cong 2 \times 10^{-16}\,\mathrm{m}$$

KK states, the KK number is conserved as an addition rule: the sum of the KK numbers of the incident particles equals the sum of the KK numbers of the outgoing particles. After the realization of this conservation law of KK numbers, one may ask the existence of some lightest stable particle. Some argues that the lightest KK state of photon might be a candidate to be classified as dark matter [7].

What do we mean by scaling down or up? For example, when we write an abelian gauge field in more than four spacetime dimensions, the coupling turns out to be dimensionful (This will be indeed observed when we write the action of a higher-dimensional gauge theory). In order to somehow tame the presumably strange behavior of gauge couplings, we need to re-scale them. The usual procedure for taming the unknown couplings is to match the theory when reduced from ED's to 4D as an effective field theory [7, 14, 15].

ADD and RS models promise to bring the cutoff scale of the SM down to the TeV scale. In both models, it is the quantum gravity that defines the cutoff scale. However, the theories of quantum gravity signal the violation of global symmetries. On the other hand, there are certain global symmetries in the SM that prevents a great many awful phenomena, including but not restricted to excessive CP violation and proton decay. Although there have appeared modification on both models, ADD and RS models are incomplete in their original form. The issue of non-violation of global symmetries seems plausible within the framework of UED model [7].

Besides its relative less reliance on a working knowledge of GR in terms of tensor calculus and its ease for better comprehension, in this project we concentrate our attention on the UED model, specifically on its minimal extension, mUED. In mUED, all the SM fields live in the bulk dimensions, whereas in a non-minimal UED theory (nmUED) fields experience localization at the boundary branes via a Dirac delta tacked in to them.

2 Methodology

2.1 Conventions and definitions

In our analysis, we abide by the following:

1. As it has been standardized in the literature of high-energy physics, we set $\hbar = c = 1$.

2. Our metric convention is *mostly negative*: $g_{\mu\nu} \doteq \mathrm{diag}(+ - - -)$.

3. Our sign convention for the covariant derivative is as in Reference [16][4]:

$$\mathscr{D}_\mu = \partial_\mu - ig_c T^i_C G^i_\mu - ig_w T^a_W W^a_\mu + ig_y T_Y B_\mu$$

4. The various Lagrangians will be defined in 5D. Instead of explicitly showing the coordinate-dependencies, we denote the effective Lagrangians by an arrow with the integration operator above it:

$$\mathscr{L} \xrightarrow{\int_0^{2\pi R} dy} \cdots$$

5. In 5D theory, we will denote the extra dimensions by be expanding the fields in a Fourier series as in

$$\mathcal{K}^+(x,y) = \frac{1}{\sqrt{2\pi R}}\mathcal{K}^+_{(0)}(x) + \sum_{n\geq 1} \frac{1}{\sqrt{\pi R}}\mathcal{K}^+_{(n)}(x) \cos\frac{ny}{R}$$

[4]The author cited in [16] published a paper on the Feynman rules in 4D by parametrizing even the signs of the gauge couplings in the covariant derivative; to wit, `arXiv:1209.6213`.

$$\mathcal{K}^-(x,y) = \sum_{n\geq 1} \frac{1}{\sqrt{\pi R}} \mathcal{K}^-_{(n)}(x) \sin\frac{ny}{R}$$

for some scalar, vector, or fermion field $\mathcal{K}$, so as to be able to obtain an effective theory for 4D theory. The fields are expanded in sines or cosines according to their parity under Z_2, which simply sends $y \to -y$. We will refer to the infinite number of states as the *Kaluza-Klein (KK) tower*. The summation index will be called the *KK number*.

2.2 A brief summary of the Standard Model: the $U(1)$ group

This section will serve as an exercise to a simple analysis of the Standard Model in four dimensions. Here, we will recall the basics of different types of Lagrangians, the extraction of the so-called Feynman rules, and last but not least a clear application of the Higgs mechanism[16, 17, 18]. The generalization to the $SU(N)$ group is straightforward, albeit more complicated mainly due to the existence of $N^2 - 1$ gauge bosons.

Let us consider an abelian 4D theory with a fermion and a Higgs scalar, mediated by an abelian gauge boson. The Lagrangian for this theory reads

$$\mathscr{L} = \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{higgs}} + \mathscr{L}_{\text{fermion}} + \mathscr{L}_{\text{yukawa}} + \mathscr{L}_{\text{gf}} + \mathscr{L}_{\text{ghosts}}$$

In an abelian theory, the Fadeev-Popov ghosts induced by the gauge-fixing Lagrangian do not interact with the matter fields and therefore the ghosts Lagrangian may be dropped. This is the case when we refer to the generalized R_ξ gauge class. In the unitary gauge, we have no ghosts; however, the unitary gauge has its own issues.

We start the theory with massless fields. After spontaneous symmetry breaking of the Higgs potential, the gauge boson and the fermion field will attain masses. The fermion field receives a mass term via a Yukawa interaction with the Higgs field; the gauge boson will do so via the kinetic term of the Higgs field. We may leave the gauge-fixing Lagrangian for later.

Let us begin our discussion by writing down the individual massless Lagrangians for the above-mentioned fields. The Proca Lagrangian describes the gauge boson; the fermion will be described by the Dirac Lagrangian:

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2$$

$$\mathscr{L}_{\text{fermion}} = \bar{\psi} i\gamma^\mu \mathscr{D}_\mu \psi$$

The Higgs field is described as usual:

$$\mathscr{L}_{\text{higgs}} = \left|\mathscr{D}_\mu H\right|^2 + \mu^2 \left|H\right|^2 - \lambda \left|H\right|^4$$

The covariant derivative is given as

$$\mathscr{D}_\mu = \partial_\mu + ieA_\mu$$

where e is the electric charge in natural units.

In general, the Lagrangian is the difference between the kinetic term and the potential term:

$$\mathscr{L} = \mathscr{T} - \mathscr{U}$$

Then we see that

$$\mathscr{U}_{\text{higgs}} = -\mu^2 \left|H\right|^2 + \lambda \left|H\right|^4$$

The Higgs field has two degrees of freedom here. If we write

$$H = H_1 + iH_2$$

for the real fields $H_{1,2}$, then we see that the potential is minimum on a circle defined as

$$\langle H_1 \rangle^2 + \langle H_2 \rangle^2 = \frac{\mu^2}{2\lambda}$$

We may choose the vacuum expectation values (VEVs) as

$$\langle H_1 \rangle = \frac{v}{\sqrt{2}}$$

$$\langle H_2 \rangle = 0$$

so $v = \sqrt{\mu^2/\lambda}$. As usual, we expand $H_{1,2}$ around their VEVs:

$$H_1 = \frac{v+h}{\sqrt{2}}$$

$$H_2 = \frac{0+\chi}{\sqrt{2}} = \frac{\chi}{\sqrt{2}}$$

and then

$$H = \frac{v+h+i\chi}{\sqrt{2}}$$

Now let us rewrite the kinetic term for the Higgs field:

$$\begin{aligned}
|D_\mu H|^2 &= \left[(\partial_\mu + ieA_\mu) \frac{v+h+i\chi}{\sqrt{2}} \right]^* \left[(\partial_\mu + ieA_\mu) \frac{v+h+i\chi}{\sqrt{2}} \right] \\
&= \left(\frac{1}{\sqrt{2}} \partial_\mu h - \frac{i}{\sqrt{2}} \partial_\mu \chi - \frac{iev}{\sqrt{2}} A_\mu - \frac{ie}{\sqrt{2}} h A_\mu - \frac{e}{\sqrt{2}} A_\mu \chi \right) \\
&\qquad \times \left(\frac{1}{\sqrt{2}} \partial_\mu h - \frac{i}{\sqrt{2}} \partial_\mu \chi - \frac{iev}{\sqrt{2}} A_\mu - \frac{ie}{\sqrt{2}} h A_\mu - \frac{e}{\sqrt{2}} A_\mu \chi \right) \\
&= \left(\frac{1}{\sqrt{2}} \partial_\mu h - \frac{e}{\sqrt{2}} A_\mu \chi \right)^2 + \left(\frac{1}{\sqrt{2}} \partial_\mu \chi + \frac{ev}{\sqrt{2}} A_\mu + \frac{e}{\sqrt{2}} h A_\mu \right)^2 \\
&= \frac{1}{2} (\partial_\mu h)^2 + \frac{1}{2} (\partial_\mu \chi)^2 + ehA_\mu \partial^\mu \chi + \frac{e^2}{2} h^2 (A_\mu)^2 + evA_\mu \partial^\mu \chi + e^2 vh (A_\mu)^2 \\
&\qquad + \frac{e^2 v^2}{2} (A_\mu)^2 - e\chi A_\mu \partial^\mu h + \frac{e^2}{2} (A_\mu)^2 \chi^2
\end{aligned}$$

The potential term for the Higgs field can also be expanded:

$$\begin{aligned}
\mathscr{U}_{\text{higgs}} &= -\mu^2 |H|^2 + \lambda |H|^4 \\
&= -\mu^2 \left| \frac{v+h+i\chi}{\sqrt{2}} \right|^2 + \lambda \left| \frac{v+h+i\chi}{\sqrt{2}} \right|^4 \\
&= \mu^2 h^2 + \frac{\mu^2}{4v^2} h^4 + \frac{\mu^2}{v} h^3 + \frac{\mu^2}{2v^2} h^2 \chi^2 + \frac{\mu^2}{v} h\chi^2 + \frac{\mu^2}{4v^2} \chi^4
\end{aligned}$$

Hence the Lagrangian for the Higgs field becomes

$$\begin{aligned}\mathscr{L}_{\text{higgs}} = &\left\{\frac{1}{2}\left(\partial_\mu h\right)^2 - \frac{1}{2}\left(\sqrt{2}\mu\right)^2 h^2\right\} + \left\{\frac{1}{2}\left(\partial_\mu \chi\right)^2\right\} + \left\{\frac{1}{2}\left(ev\right)^2\left(A_\mu\right)^2\right\} \\ &+ \left\{\frac{1}{2}e^2h^2\left(A_\mu\right)^2 + e^2vh\left(A_\mu\right)^2\right\} + \left\{\frac{\mu^2}{4v^2}h^4 + \frac{\mu^2}{v}h^3\right\} \\ &+ \left\{ehA_\mu\partial^\mu\chi - e\chi A_\mu\partial^\mu h\right\} + \left\{evA_\mu\partial^\mu\chi + \frac{1}{2}e^2\left(A_\mu\right)^2\chi^2\right\} \\ &+ \left\{\frac{\mu^2}{2v^2}h^2\chi^2 + \frac{\mu^2}{v}h\chi^2\right\} + \left\{\frac{\mu^2}{4v^2}\chi^4\right\}\end{aligned}$$

If we consider the massive Klein-Gordon Lagrangian for the Higgs field, we see that the mass of the Higgs boson is

$$m_h = \sqrt{2}\mu$$

If we consider the massive Proca Lagrangian for the gauge field, we see that the mass of the gauge boson is

$$m_A = ev$$

In addition to the mass terms, we have the interactions $hh\gamma$, $hh\gamma\gamma$, $hhhh$, hhh, and interaction of h and γ with the Goldstone boson χ.

In closing up this section, we finally consider the Yukawa Lagrangian:

$$\mathscr{L}_{\text{yukawa}} = -y\bar{\psi}\psi H$$

In this toy model, we have only one fermion, and hence the Yukawa coupling is indeed a real constant, and we do not have a Hermitian conjugate tail.

$$\begin{aligned}\mathscr{L}_{\text{yukawa}} &= -y\bar{\psi}\psi H \\ &= -y\bar{\psi}\psi\frac{v+h+i\chi}{\sqrt{2}} \\ &= -\frac{yv}{\sqrt{2}}\bar{\psi}\psi - \frac{y}{\sqrt{2}}h\bar{\psi}\psi - \frac{iy}{\sqrt{2}}\chi\bar{\psi}\psi\end{aligned}$$

The first term gives the fermion mass:

$$m_\psi = \frac{yv}{\sqrt{2}}$$

The others give interaction of the fermion with the Higgs and the Goldstone bosons, respectively. Clearly, in a more complicated theory such as one with an $SU(N)$ symmetry, we expect things to become complicated: when the multiplets of fermions begin emerging, the Yukawa coupling will no longer be a simple scalar but a matrix of the size of the multiplet. In that case, the usual procedure is to diagonalize the Yukawa matrices. In the real-world 4D theory of the Standard Model, we have the quark mixing, as well, which requires paying even more attention.

2.3 Promoting SM to 5D

In 5D, our universe is depicted what Theodor Kaluza would define as a cylindrical one (Figure 1).

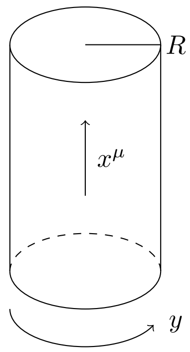


Figure 1: The geometry of a 5D world with the extra dimensions compactified on a circle of some small radius R.

In promoting the Standard Model to the five-dimensional (5D) case[8, 15, 16, 19, 20, 21, 22, 23, 24][5], we merely

1. let the Lorentz indices take on an additional value, 5, and denote the new spatial component by $y := x^5$, and

2. make a harmonic expansion of the fields in the new spatial component so as to be able to integrate out the fifth component easily.

For the first item, we write

$$\begin{aligned} \mu = 0,1,2,3 \quad &\longrightarrow \quad M = 0,1,2,3,5 \\ x^\mu = \left(x^0, \vec{x}\right) \quad &\longrightarrow \quad x^M = \left(x^0, \vec{x}, y\right) \\ \partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad &\longrightarrow \quad \partial_M = \left(\frac{\partial}{\partial t}, \vec{\nabla}, \frac{\partial}{\partial y}\right) \end{aligned}$$

For the second item, care must be provided in assigning the parity of the fields under Z_2 symmetry, which just says $y \to -y$.

As for the usual constants of SM, we tack in a subscript 5:

$$\begin{aligned} g_c \quad &\longrightarrow \quad g_{c5} \\ g_w \quad &\longrightarrow \quad g_{w5} \\ g_y \quad &\longrightarrow \quad g_{y5} \end{aligned}$$

where g_c, g_w, and g_y denote the coupling of the color charge, the weak charge, and the hypercharge, respectively, in 4D.

To simplify the notation, we suppress the coordinate dependencies of the field until a further notice.

2.4 The complete SM Lagrangian

The complete Lagrangian for the Standard Model is given as usual:

$$\mathscr{L} = \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{higgs}} + \mathscr{L}_{\text{fermion}} + \mathscr{L}_{\text{yukawa}} + \mathscr{L}_{\text{gf}} + \mathscr{L}_{\text{ghost}}$$

[5] We borrow the complete Lagrangian for the Standard Model in 4D from Reference [16] with a consistent set of signs, and apply the two-item procedure as described in the other references just cited but in our notation.

All the fields (gauge, Higgs, and matter) start out massless, therefore only the kinetic term for each expression should be kept. Thus, we are expected to collect only these terms in the usual Lagrangians:

$$\mathscr{L}_{\text{scalar}} = \frac{1}{2}\left(\mathscr{D}_M \phi\right)^2 \qquad \text{Klein-Gordon Lagrangian}$$

$$\mathscr{L}_{\text{fermion}} = \bar{\psi} i \Gamma^M \mathscr{D}_M \psi \qquad \text{Dirac Lagrangian}$$

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4}\left(V^n_{MN}\right)^2 \qquad \text{Proca Lagrangian}$$

where $\mathscr{D}_M$ is the covariant derivative in 5D and $n = 1, 2, \ldots, (N^2 - 1)$ for the symmetry groups $SU(N)$, and there will be only one n for the $U(1)$ group.

When the mass terms are added, the Lagrangians will not be invariant under the symmetry $SU(3)_C \otimes SU(2)_W \otimes U(1)_Y$, since the combined theory is non-abelian.

2.5 The gauge sector

The usual Lagrangian for the gauge sector promoted to 5D can be given as

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4}\left(G^i_{MN}\right)^2 - \frac{1}{4}\left(W^a_{MN}\right)^2 - \frac{1}{4}\left(B_{MN}\right)^2$$

2.5.1 Color charge

The first term in $\mathscr{L}_{\text{gauge}}$ refers to the field strength tensor of the gluons.

$$G^i_{MN} = \partial_M G^i_N - \partial_N G^i_M + g_{c5} f^{ijk} G^j_M G^k_N$$

where $i, j, k = 1, 2, \ldots, 8$. Here, the f^{ijk} are the structure constants that appear in the Lie algebra

$$[T^i_C, T^j_C] = i f^{ijk} T^k_C$$

where $T^i_C = \lambda^i/2$ are the generators of the group $SU(3)_C$ and the λ^i are the Gell-Mann matrices.

2.5.2 Weak charge

The second term in $\mathscr{L}_{\text{gauge}}$ refers to the field strength tensor of the gauge bosons of the weak charge.

$$W^a_{MN} = \partial_M W^a_N - \partial_N W^a_M + g_{w5} \varepsilon^{abc} W^b_M W^c_N$$

where $a, b, c = 1, 2, 3$. Here, ε^{abc} is the usual Levi-Civita pseudotensor, which is also the structure constant that appear in the Lie algebra

$$[T^a_W, T^b_W] = i \varepsilon^{abc} T^c_W$$

where $T^a_W = \tau^a/2$ are the generators of the group $SU(2)_W$ and the τ^a are the Pauli matrices.

2.5.3 Hypercharge

The last term in $\mathscr{L}_{\text{gauge}}$ refers to the field strength tensor of the gauge boson of the hypercharge.

$$B_{MN} = \partial_M B_N - \partial_N B_M$$

2.6 The Higgs sector

The usual Higgs Lagrangian promoted to 5D can be expressed as

$$\mathscr{L}_{\text{higgs}} = (\mathscr{D}_M H)^\dagger \left(\mathscr{D}^M H\right) + \mu_5 H^\dagger H - \lambda_5 \left(H^\dagger H\right)^2$$

2.6.1 The Higgs doublet

Here, H is the usual Higgs doublet:

$$H = \begin{pmatrix} \phi_+ \\ \dfrac{h + i\phi_Z}{\sqrt{2}} \end{pmatrix}$$

The vacuum expectation value (VEV) of the Higgs is hidden inside the field $h = h(x, y)$, which will become more apparent when we move on to make a harmonic expansion of the fields.

2.6.2 The covariant derivative

As in the 4D case, the covariant derive is defined in terms of gauge couplings, group generators and gauge bosons as follows:

$$\mathscr{D}_M = \partial_M - ig_{c5} T_C^i G_M^i - ig_{w5} T_W^a W_M^a + ig_{y5} T_Y B_M$$

where, as earlier,

$$T_C^i = \frac{\lambda^i}{2}, \qquad T_W^a = \frac{\tau^a}{2} \qquad T_Y = \frac{Y}{2}$$

As we know from the 4D case, the weak and hypercharge forces mix. Therefore, the physical gauge bosons are not W_M^1, W_M^2, W_M^3, and B_M but $W_M^\pm$, Z_M, and A_M:

$$W_M^\pm = \frac{W_M^1 \mp i W_M^2}{\sqrt{2}}$$

$$W_M^3 = Z_M c_w + A_M s_w$$
$$B_M = -Z_M s_w + A_M c_w$$

where c_w (s_w) is the cosine (sine) of the Weinberg angle, θ_w. In a more compact notation, this could be written as

$$\begin{pmatrix} W_M^+ \\ W_M^- \\ Z_M \\ A_M \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & & \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & & \\ & & c_w & -s_w \\ & & s_w & c_w \end{pmatrix} \begin{pmatrix} W_M^1 \\ W_M^2 \\ W_M^3 \\ B_M \end{pmatrix} \tag{2.1}$$

In this basis, the photon will remain massless after spontaneous symmetry breaking.

Let us also define the ladder operators:

$$\tau^\pm = \frac{\tau^1 \pm i\tau^2}{2},$$

We know that the gauge couplings are related to each other via the equation[6]

$$e = g_c s_w = g_y c_w$$

This should be generalized to 5D without loss of information.

Finally, we have the relation between electric charge, the third component of weak isospin, and hypercharge:

$$Q = T_W^3 + T_Y = \frac{\tau^3}{2} + \frac{Y}{2}$$

By combining what we have said so far, we can rewrite the most general form of the covariant derivative in terms of the physical gauge bosons as

$$\mathscr{D}_M = \partial_M - ig_{c5}\frac{\lambda^i}{2}G^i_M - i\frac{g_{w5}}{\sqrt{2}}(\tau^+ W^+_M + \tau^- W^-_M) - i\frac{g_{w5}}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right)Z_M + ie_5 Q A_M$$

The covariant derivative for the Higgs doublet

By using the relation

$$Q = \frac{\tau^3}{2} + \frac{Y}{2}$$

we have removed the hypercharge out of discussion and now we may continue merely with the electric charge.

The upper component of the Higgs doublet is a charged scalar, whilst the lower component is neutral. Therefore, the electric charge matrix reads

$$Q = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$$

Meantime, the Higgs field does not carry any color charge, therefore it will stay inert to the gluons. Hence the covariant derivative for the Higgs doublet reads

$$\mathscr{D}_M = \partial_M \mathbb{1}_2 - i\frac{g_{w5}}{\sqrt{2}}\begin{pmatrix} & W^+_M \\ W^-_M & \end{pmatrix} - i\frac{g_{w5}}{c_w}\begin{pmatrix} \frac{1}{2} - s_w^2 & \\ & -\frac{1}{2} \end{pmatrix} Z_M + ie_5 \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} A_M \tag{2.2}$$

where $\mathbb{1}_D$ is the D-dimensional unit matrix.

The covariant derivative for the quark triplet

We separate out the color charge of the quark from the weak charge. Thus, the quark triplet, $q = \begin{pmatrix} q(r) & q(g) & q(b) \end{pmatrix}^T$, will necessarily comprise pure quark-gluon interactions. As a result, we simply have

$$\mathscr{D}^q_M = \partial_M \mathbb{1}_3 - ig_{c5}\frac{\lambda^i}{2}G^i_M$$

[6] There is also a tacit assumption that the Weinberg angle also depends on the KK level. That is, the gauge bosons will mix with each other only within the same KK level. Thus it would be wise to denote this dependence as $s_{(n)w}$ and $c_{(n)w}$, yet the notation might get easily out of hand.

The covariant derivative for the quark doublet

Since the quark-gluon interaction has been taken care of, we now focus on the usual fermion doublet. Their structure will be the same – the ordinary covariant derivative minus the gluons. The only difference will be in the electric charge matrix. Accordingly, for the quark doublet[7] $\mathcal{Q} = \begin{pmatrix} u & d \end{pmatrix}_L^T$, we have

$$Q = \begin{pmatrix} \frac{2}{3} & \\ & -\frac{1}{3} \end{pmatrix}$$

and hence

$$\mathscr{D}_M^{\mathcal{Q}} = \partial_M \mathbb{1}_2 - i\frac{g_{w5}}{\sqrt{2}} \begin{pmatrix} & W_M^+ \\ W_M^- & \end{pmatrix} - i\frac{g_{w5}}{c_w} \begin{pmatrix} \frac{1}{2} - \frac{2}{3}s_w^2 & \\ & -\frac{1}{2} + \frac{1}{3}s_w^2 \end{pmatrix} Z_M + ie_5 \begin{pmatrix} \frac{2}{3} & \\ & -\frac{1}{3} \end{pmatrix} A_M$$

The covariant derivative for the up-type quark singlet

Both lepton and quark singlets will not interact with the charged weak gauge boson, $W_M^\pm$. This can be reflected mathematically by canceling out the Pauli matrices out of the covariant derivative. For the up-type quark singlet, by also noting that $Q = 2/3$, we have

$$\mathscr{D}_M^u = \partial_M + i\frac{g_{w5}}{c_w}\frac{2}{3}s_w^2 Z_M + ie_5\frac{2}{3}A_M$$

The covariant derivative for the down-type quark singlet

This differs from the covariant derivative for the up-type quark singlet merely in charge. With $Q = -1/3$, we have

$$\mathscr{D}_M^d = \partial_M - i\frac{g_{w5}}{c_w}\frac{1}{3}s_w^2 Z_M - ie_5\frac{1}{3}A_M \tag{2.3}$$

The covariant derivative for the lepton doublet

This is identical to the covariant derivative for the quark doublet, except for the charges. We now have the charge matrix $Q = \mathrm{diag}(0, -1)$, and therefore the covariant derivative for the doublet $\mathcal{L} = \begin{pmatrix} \nu_l & l \end{pmatrix}_L^T$ reads

$$\mathscr{D}_M^{\mathcal{L}} = \partial_M \mathbb{1}_2 - i\frac{g_{w5}}{\sqrt{2}} \begin{pmatrix} & W_M^+ \\ W_M^- & \end{pmatrix} - i\frac{g_{w5}}{c_w} \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} + s_w^2 \end{pmatrix} Z_M + ie_5 \begin{pmatrix} 0 & \\ & -1 \end{pmatrix} A_M$$

The covariant derivative for the lepton singlet

This is almost identical to the covariant derivative for the quark singlet, with $Q = -1$:

$$\mathscr{D}_M^l = \partial_M - i\frac{g_{w5}}{c_w}s_w^2 Z_M - ie_5 A_M \tag{2.4}$$

[7] We denote a generic up-type quark by u and a down-type quark by d.

2.7 The fermion sector

Our way of expressing the fermion Lagrangian is as follows:

$$\mathscr{L}_{\text{fermion}} = \bar{q}i\Gamma^M \mathscr{D}^q_M q + \bar{\mathcal{Q}}i\Gamma^M \mathscr{D}^{\mathcal{Q}}_M \mathcal{Q} + \bar{u}i\Gamma^M \mathscr{D}^u_M u + \bar{d}i\Gamma^M \mathscr{D}^d_M d + \bar{\mathcal{L}}i\Gamma^M \mathscr{D}^{\mathcal{L}}_M \mathcal{L} + \bar{l}i\Gamma^M \mathscr{D}^e_M l$$

where

$$q = \text{quark triplet} = \begin{pmatrix} q(r) & q(g) & q(b) \end{pmatrix}^T$$

$$\mathcal{Q} = \text{quark doublet} = \begin{pmatrix} u & d \end{pmatrix}^T_L$$

$$u = \text{up-type quark singlet} = u_R$$

$$d = \text{down-type quark singlet} = d_R$$

$$\mathcal{L} = \text{lepton doublet} = \begin{pmatrix} \nu_l & l \end{pmatrix}^T_L$$

$$l = \text{electron-type lepton singlet} = l_R$$

and sum over generations is implicit. The 5D Dirac matrices meantime are modified as

$$\Gamma^M = \left(\gamma^\mu, i\gamma^5\right)$$

The Clifford algebra remains its form:

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}$$

where g_{MN} is the *mostly negative* metric in 5D:

$$g_{MN} \doteq \begin{pmatrix} g_{\mu\nu} & \\ & -1 \end{pmatrix}$$

2.8 The Yukawa sector

The Yukawa Lagrangian, which not only describes the interaction of the fermions with the charged and neutral scalars but also generates the fermion masses after spontaneous symmetry breaking via a Yukawa interaction with the Higgs field, is written in terms of non-diagonalized Yukawa matrices as

$$\mathscr{L}_{\text{yukawa}} = -y_{d5}\bar{\mathcal{Q}}dH - y_{u5}\bar{\mathcal{Q}}u\tilde{H} - y_{l5}\bar{\mathcal{L}}lH + \text{h.c.}$$

where

$$\tilde{H} = i\tau^2 H^* = \begin{pmatrix} \dfrac{h - i\phi_Z}{\sqrt{2}} \\ \phi_- \end{pmatrix}$$

is the charge conjugate of the Higgs doublet, H, and the y_{j5} are the Yukawa interaction matrices.

2.9 The gauge-fixing Lagrangian and the ghosts

To a researcher candidate in high-energy physics, the gauge-fixing Lagrangian is at the very heart of an analysis of SM, whether in 4D or in 5D. So far in introductory particle physics courses, we have become familiar with working in the unitary gauge (UG) [17]. UG has its traits. To begin with, we witness that the Higgs doublet takes the simplest possible form

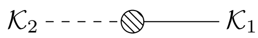
$$H_{\mathrm{UG}} = \begin{pmatrix} 0 \\ \frac{h}{\sqrt{2}} \end{pmatrix}$$

where the Higgs VEV is encoded in the expansion of the field h. Secondly, there will be no unphysical particles, or ghosts. The scalars introduced in the gauge-fixing Lagrangian will be redundant and the theory will partly become much simpler to study. Finally, the seemingly interaction terms of the form

$$\mathcal{K}_1 \partial_\mu \mathcal{K}_2$$

where, for instance, $\mathcal{K}_2$ may be one of those scalars introduced in the gauge-fixing Lagrangian in the R_ξ gauge class, will also be removed out. In a way, UG works just fine at the tree level. From introductory courses, we know that there exist loop corrections to the masses and gauge couplings. This is where the unitary gauge ceases being useful: the propagators will be proportional to inverse mass squared, which is a dimensionful quantity, which in turn makes the theory non-renormalizable. To raze off this effect, it is quite beneficial to switch to the Feynman-'t Hooft gauge. Basically, it is just a special case of the more general R_ξ gauge class, for which the theory was shown to renormalizable by Gerardus 't Hooft himself.

From a preliminary experience with the 4D theory in the Feynman-'t Hooft gauge, we know the calculations may easily prove to be cumbersome. The first reason is the existence of the terms as described above – a field times the 4-gradient of another field. If, for example, $\mathcal{K}_1$ is a vector field and $\mathcal{K}_2$ is a scalar and if we were to depict this term with a Feynman diagram, we would draw



What is even worse is that we somehow have to live with this: a scalar turns into a vector for no apparent reason as it propagates. Thus we infer that the R_ξ gauge class in general induces a ghost Lagrangian that will cancel out such vertices.

2.10 The essentials

The extra dimension is compactified on a circle with an interchange symmetry, that is, the compactification is S^1/Z_2, and this implies a parity. Since the fifth component is periodic, we may perform a harmonic expansion of the fields along the fifth dimensions. By *harmonic expansion*, we clearly mean the Fourier series. Let us revise the basics of the Fourier series pertaining to our discussion.

Any *nice enough* function can be expanded in a Fourier series as usual:

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} (a_n \cos nx + b_n \sin nx)$$

If there exists a symmetry under $x \to -x$, we should indicate this explicitly as

$$f(x) = \pm f(-x) \qquad \text{or} \qquad f_+(x) = +f_+(-x), \quad f_-(x) = -f_-(-x)$$

For even parity, we should drop the sine term:

$$f_+(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos nx$$

For odd parity, we consider only the sine term:

$$f_-(x) = \sum_{n\geq 1} b_n \sin nx$$

Long story short, we should assign parity to the fields correctly. To illustrate, the correct way to do this for the photon field will be as

$$A_\mu(x,y) = +A_\mu(x,-y)$$
$$A_5(x,y) = -A_5(x,-y)$$

There is a physical reasoning behind this (as it will the case for all the other fields): we should recover $A_{(0)\mu}$, that is, we should not give up on the original photon. Clearly, we do not have a scalar A_5 in 4D, so it should not emerge in the zeroth mode. Under this considerations, and by defining

$$c_n := \frac{1}{\sqrt{\pi R}} \cos \frac{ny}{R}$$
$$s_n := \frac{1}{\sqrt{\pi R}} \sin \frac{ny}{R}$$

we have the following expansions[8]

$$V_\mu = \frac{1}{\sqrt{2\pi R}} V_{(0)\mu} + V_{(n)\mu} c_n$$
$$V_5 = V_{(n)5} s_n$$
$$\Phi = \frac{1}{\sqrt{2\pi R}} \Phi_{(0)} + \Phi_{(n)} c_n$$
$$\mathcal{F} = \frac{1}{\sqrt{2\pi R}} \mathcal{F}_{(0)} + P_L \mathcal{F}_{(n)} c_n + P_R \mathcal{F}_{(n)} s_n$$
$$f = \frac{1}{\sqrt{2\pi R}} f_{(0)} + P_R f_{(n)} c_n + P_L f_{(n)} s_n$$

where

$$V_M = \{G^i_M, W^\pm_M, Z_M, A_M\}$$

[8] To facilitate typing, let us suppress the coordinate dependencies: if a field does not have a KK index such as (j), then it will be a function of both the usual 4-position and the extra dimension. Also, let us suppress the summation symbols by employing the Einstein summation convention. For example, for the photon field, we explicitly have

$$A_\mu(x,y) = \frac{1}{\sqrt{2\pi R}} A_{(0)\mu}(x) + \sum_{n\geq 1} \frac{1}{\sqrt{\pi R}} A_{(n)\mu}(x) \cos \frac{ny}{R}$$
$$A_5(x,y) = \sum_{n\geq 1} \frac{1}{\sqrt{\pi R}} A_{(n)5}(x) \sin \frac{ny}{R}$$

$$\Phi = \{\phi_\pm, \phi_Z, h\}$$
$$\mathcal{F} = \{\mathcal{Q}, \mathcal{L}\}$$
$$f = \{u, d, l\}$$

and

$$P_{R/L} = \text{projection operators} = \frac{1 \pm \gamma^5}{2}$$

Here, we note that

1. the coefficients $1/\sqrt{2\pi R}$ and $1/\sqrt{\pi R}$ are chosen so as to comply with the canonical normalization.
2. all the scalars we *do* have in the 4D model should be recovered, therefore we should assign the scalars a positive KK parity.
3. only for the Higgs scalar, the zeroth term will get a non-vanishing vacuum expectation value:
$$h_{(0)} \longrightarrow h_{(0)} + v_5$$
4. the fermion expansion may look strage at first, but what they represent makes perfect sense. First of all, we see that the original left- (L-) and right-handed (RH) fermions are recovered in the zeroth mode. Furthermore, in the extra dimension, we get additional chirality: for a LH fermion, there appears both a LH (by applying P_L) and a RH fermion in the extra dimension, which might be thought of as LLH and LRH. The specific choice of the factor $P_{R/L}$ is again due to the recovery of the original particles in the zeroth mode. As we can see in the expansion, the LLH component survives in the zeroth mode if we put $n = 0$ in the expansion. There is no LRH fermion in 4D, which could possibly be obtained solely by the operation $P_R \psi_L$ in the real world, but there might be one in the extra dimension.

2.10.1 Orthogonality of many sines and cosines

In 4D, the Higgs Lagrangian may be exploded to yield a zoo of interaction terms – interaction of three or four fields. Furthermore, there always appear mass terms for the fields – *interaction* of a field with itself. There may emerge also terms made up of a scalar times a field. If we combine this observation with the exercise of expanding the fields with sines and cosines, it will be inevitable to evaluate some orthogonality integrals of pure cosines, pure sines, and mixtures of them. The ones that we could catch so far include the following:

$$\Delta_n := \int_0^{2\pi R} dy\, c_n = \int_0^{2\pi R} dy\, s_n$$

$$\Delta_{nm} := \int_0^{2\pi R} dy\, c_n c_m = \int_0^{2\pi R} dy\, s_n s_m$$

$$\Delta'_{nm} := \int_0^{2\pi R} dy\, c_n s_m$$

$$\Delta_{nmk} := \int_0^{2\pi R} dy\, c_n c_m c_k$$

$$\Delta_{nm,k} := \int_0^{2\pi R} dy\, s_n s_m c_k$$

$$\Delta_{n,mk} := \int_0^{2\pi R} dy\, s_n c_m c_k$$

$$\Delta'_{nmk} := \int_0^{2\pi R} dy\, s_n s_m s_k$$

$$\Delta_{nmkl} := \int_0^{2\pi R} dy\, c_n c_m c_k c_l$$

$$\Delta_{nm,kl} := \int_0^{2\pi R} dy\, s_n s_m c_k c_l$$

with

$$c_n = \frac{1}{\sqrt{\pi R}} \cos\frac{ny}{R}, \qquad s_n = \frac{1}{\sqrt{\pi R}} \sin\frac{ny}{R}$$

Let us first evaluate the integrals that yield a zero: Over its one period, any sine or cosine function vanishes:

$$\Delta_n = 0$$

Another trivial integral is the product of sine and cosine:

$$\Delta'_{nm} = 0$$

The third one is the double cosines and a sine. By using MATHEMATICA, we get

$$\begin{aligned}
\Delta_{n,mk} &= \frac{1}{4\pi^{3/2}\sqrt{R}} \left(\frac{1}{k-m-n} + \frac{1}{k+m-n} + \frac{1}{k-m+n} + \frac{1}{k+m+n} \right. \\
&\quad - \frac{\cos 2\pi(k-m-n)}{k-m-n} - \frac{\cos 2\pi(k+m-n)}{k+m-n} - \frac{\cos 2\pi(k-m+n)}{k-m+n} \\
&\quad \left. - \frac{\cos 2\pi(k+m+n)}{k+m+n} \right) \\
&= \frac{1}{4\pi^{3/2}\sqrt{R}} \left(\frac{1}{k-m-n} + \frac{1}{k+m-n} + \frac{1}{k-m+n} + \frac{1}{k+m+n} \right. \\
&\quad \left. - \frac{1}{k-m-n} - \frac{1}{k+m-n} - \frac{1}{k-m+n} - \frac{1}{k+m+n} \right) \\
&= 0
\end{aligned}$$

Yet another one that gives zero is the triple sines. The MATHEMATICA output of this integral is

$$\begin{aligned}
\Delta'_{nmk} &= \frac{1}{4\pi^{3/2}\sqrt{R}}\left(\frac{1}{k+m-n}+\frac{1}{k-m+n}+\frac{1}{-k+m+n}-\frac{1}{k+m+n}\right.\\
&\quad+\frac{\cos 2\pi(k-m-n)}{k-m-n}-\frac{\cos 2\pi(k+m-n)}{k+m-n}-\frac{\cos 2\pi(k-m+n)}{k-m+n}\\
&\quad\left.+\frac{\cos 2\pi(k+m+n)}{k+m+n}\right)\\
&= \frac{1}{4\pi^{3/2}\sqrt{R}}\left(\frac{1}{k+m-n}+\frac{1}{k-m+n}+\frac{1}{-k+m+n}-\frac{1}{k+m+n}\right.\\
&\quad\left.+\frac{1}{k-m-n}-\frac{1}{k+m-n}-\frac{1}{k-m+n}+\frac{1}{k+m+n}\right)\\
&= 0
\end{aligned}$$

For the rest of the terms, we continue exploiting the computational power of MATHEMATICA:

$$\Delta_{nm} = \frac{1}{2\pi}\left(\frac{\sin 2\pi(m-n)}{m-n}+\frac{\sin 2\pi(m+n)}{m+n}\right)$$

Since $n, m \in \mathbb{Z}^+$, the second term is always zero. In the limit $m \to n$, the first term gives 2π, which is otherwise zero. Thus we have[9]

$$\Delta_{nm} = \frac{1}{2\pi}\left[2\pi\delta(m-n)\right] = \delta(m-n)$$

By the same token, the triple integration

$$\Delta_{nmk} = \frac{1}{4\pi^{3/2}R^{1/2}}\left(\frac{\sin 2\pi(n-m-k)}{n-m-k}+\frac{\sin 2\pi(n-m+k)}{n-m+k}+\frac{\sin 2\pi(n+m-k)}{n+m-k}\right.$$
$$\left.+\frac{\sin 2\pi(n+m+k)}{n+m+k}\right)$$

can be reduced to a collection of delta functions as

$$\begin{aligned}
\Delta_{nmk} &= \frac{1}{4\pi^{3/2}R^{1/2}}\left[2\pi\delta(n-m-k)+2\pi\delta(n-m+k)+2\pi\delta(n+m-k)\right]\\
&= \frac{1}{2\sqrt{\pi R}}\left[\delta(n-m-k)+\delta(n-m+k)+\delta(n+m-k)\right]
\end{aligned}$$

Let us recall that these integrals are involved in the vertex factors, so they will describe the conservation of momentum. In the case of the double cosines, we only have the option $n \longrightarrow n$ or vice versa, that is, the particle with a KK index n should go to a particle with an index n. The sine term we dropped would cause a vertex like $n+m \longrightarrow 0$, which is mere nonsense. Next,

[9] Instead of the Kronecker delta, we make use of the Dirac delta function on the integer domain. This way, we recover a notation which is employed in describing the vertex factors in the usual set of Feynman rules in 4D, for which we also observe the rule *each δ gets a 2π*. After all, the integers $n, m, k, l, \ldots$ are nothing but the eigenvalues of the momentum in the fifth dimension, up to factor of R. Thus, it is purely a choice for us to use the *good old* Dirac delta notation to denote the conservation of momentum at each vertex.

in the case of triple cosines, we have more options of combining three indices: $n \longrightarrow m + k$ or vice versa (since the Dirac delta is an even function of its argument), in cyclic. Therefore, we may have the ansatz for the quadruple cosines: it will describe the vertices in the form

$$n \longrightarrow m + k + l \qquad \text{(cyclic)}$$

$$n + m \longrightarrow k + l \qquad \text{(cyclic)}$$

There will be $2^3 - 1$ Dirac delta functions, accordingly (2 for two options, + or $-$; 3 for three 'white spaces' between the KK indices, $n\ m\ k\ l$; -1 for eliminating the combination $n+m+k+l$); the overall factor can again be found by using MATHEMATICA:

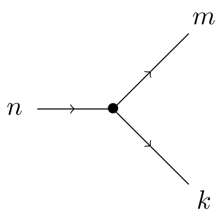
$$\begin{aligned}\Delta_{nmkl} = \frac{1}{8\pi^2 R}[&2\pi\delta(n+m+k-l) + 2\pi\delta(n+m-k+l) \\ &+ 2\pi\delta(n+m-k-l) + 2\pi\delta(n-m+k+l) \\ &+ 2\pi\delta(n-m+k-l) + 2\pi\delta(n-m-k+l) \\ &+ 2\pi\delta(n-m-k-l)]\end{aligned}$$

The vertex factors that will arise due to these Dirac deltas may be summarized as follows:

1. Δ_n: $n \longrightarrow ?$ – particle goes in but does not come out, so it is nonsense.

2. Δ_{nm}: $n \longrightarrow n$ – particles goes in and the same particle comes out, so it is probably a mass term if there appears no field with a KK index 0.

3. Δ_{nmk}: $n \longrightarrow m + k$, or vice versa, in cyclic – if there is not a field with a zero KK index, this is a vertex of three KK fields.



(cyclic)

4. Δ_{nmkl}: $n \longrightarrow m + k + l$ or $n + m \longrightarrow k + l$ or vice versa, in cyclic – these depict vertices with four fields (since the highest power in any field in all of the Lagrangians is 4, this is the *largest* vertex we will ever see).

The integrals with sines are no different, except for the sign of certain Dirac deltas. We may summarize the results as follows:

$$\Delta_n = 0$$

$$\Delta_{nm} = \delta(n - m)$$

$$
\begin{aligned}
\Delta'_{nm} &= 0 \\
\Delta_{nmk} &= \frac{1}{2\sqrt{\pi R}}[\delta(n-m-k)+\delta(n-m+k)+\delta(n+m-k)] \\
\Delta_{nm,k} &= \frac{1}{2\sqrt{\pi R}}[\delta(m-n-k)-\delta(m+n-r)+\delta(m-n+k)] \\
\Delta_{n,mk} &= 0 \\
\Delta_{nmkl} &= \frac{1}{4\pi R}[\delta(n+m+k-l)+\delta(n-m+k+l)+\delta(n+m-k-l) \\
&\qquad\qquad + \delta(n-m+k+l)+\delta(n-m+k-l)+\delta(n-m-k+l) \\
&\qquad\qquad + \delta(n-m-k-l)] \\
\Delta_{nm,k} &= \frac{1}{2\sqrt{\pi R}}[\delta(m-n-k)-\delta(m+n-r)+\delta(m-n+k)] \\
\Delta'_{nmk} &= 0 \\
\Delta_{nm,kl} &= \frac{1}{4\pi R}[\delta(m-n-k-l)-\delta(m+n-k-l)+\delta(m-n+k-l) \\
&\qquad\qquad - \delta(m+n+k-l)+\delta(m-n-k+l)-\delta(m+n-k+l) \\
&\qquad\qquad + \delta(m-n+k+l)]
\end{aligned}
$$

The most fundamental implication of these integrals rests in the conservation of KK number.

2.10.2 The KK parity

There is one final issue to mention about when it comes to the extra dimension. Although the integrals in the previous section offer a nice set of selection rules, this is not the whole story. We have imposed a parity, Z_2, on the circle. Accordingly, consider a translation in the fifth dimension by an amount πR [7, 8, 25]:

$$
\begin{aligned}
\cos\frac{ny}{R} \longrightarrow \cos\frac{n}{R}(y+\pi R) &= \cos\frac{ny}{R}\cos n\pi - \sin\frac{ny}{R}\sin n\pi \\
&= (-)^n \cos\frac{ny}{R}
\end{aligned}
$$

$$
\begin{aligned}
\sin\frac{ny}{R} \longrightarrow \sin\frac{n}{R}(y+\pi R) &= \sin\frac{ny}{R}\cos n\pi + \cos\frac{ny}{R}\sin n\pi \\
&= (-)^n \sin\frac{ny}{R}
\end{aligned}
$$

Therefore, we have just discovered a new symmetry, and each KK state has now attained a parity – the KK parity, $(-)^n$. To see its implications, consider the vertex $n \longrightarrow m+k$. The KK parity in the initial state is $(-)^n$. In the final state, the KK parities are $(-)^m$ and $(-)^k$. By conservation of KK numbers, we have $n = m+k$; thus, the KK parity is also conserved if it is a multiplicative quantum number:

$$(-)^n = (-)^{m+k}$$

The conservation of KK parity has implications (Kribs): First of all, the lightest KK state ($n = 1$) will not exhibit any decay mode. Thus, the first KK state of all the fields are stable. If the field is also electrically neutral, that field may prove to be a candidate for dark matter. One such field that comes to mind is the photon. Secondly, the KK states with odd parity will be produced in pairs. This is a heavy constraint on the lighter one imposed by the Dirac deltas, for instance, $n \longrightarrow m+k$.

3 Sample calculations

Once we have promoted the fields and accordingly the related Lagrangians to 5D, the rest is a cute exercise of expanding the terms and integrating over the fifth component, y, on its domain $[0, 2\pi R]$. In the literature, there are some results of some vertex factors and examples of implementations of the whole theory on MATHEMATICA. Therefore, what we plan to do is simply try to reproduce some parts to see whether there are differences with the literature results.

In expanding the fields, calculating the products, and performing the integrals, we will heavily rely on a working knowledge of MATHEMATICA, that is, we will be implementing part of our model, whether it means in a way *rediscovering the America* or not.

3.1 The gauge sector

The masses of the gauge bosons are partly encoded within their kinetic terms and partly within the Higgs Lagrangian. Usually in 4D, we would need the kinetic terms only for the gauge propagators and gauge interactions; however, when we promote the theory to 5D and integrate out the extra dimension, we will recover the 4D in the zeroth mode in addition to some mass terms for the KK towers of the gauge fields.

For the sake of simplicity, let us ignore the gluons for the moment. The gauge Lagrangian for the electroweak interaction is given as

$$\mathscr{L}^{\mathrm{EW}}_{\mathrm{gauge}} = -\frac{1}{4}\left(W^a_{MN}\right)^2 - \frac{1}{4}\left(B_{MN}\right)^2$$

However, we cannot directly work with W^a_{MN} and B_{MN}; we have to switch to the physical basis. There are two options at this point: either we explicitly perform the summation over a, add the hypercharge boson to it, and convert the fields into the physical fields according to (2.1), or we can be smart to begin with and collect the certain terms to form nice-looking combinations. We proceed with the latter.

To begin with, let us define

$$W^{\pm}_{MN} := \frac{1}{\sqrt{2}}\left(W^1_{MN} \mp iW^2_{MN}\right)$$

inspired by the definition of the $W^{\pm}$ bosons out of $W^{1,2}$:

$$\begin{aligned}
W^{+}_{MN} &= \frac{1}{\sqrt{2}}[\partial_M W^1_N - \partial_N W^1_M + g_{w5}\varepsilon^{1bc}W^b_M W^c_N \\
&\qquad\qquad - i(\partial_M W^2_N - \partial_N W^2_M + g_{w5}\varepsilon^{1de}W^d_M W^e_N)] \\
&= \frac{1}{\sqrt{2}}[\partial_M W^1_N - \partial_N W^1_M + g_{w5}(W^2_M W^3_N - W^3_M W^2_N) \\
&\qquad\qquad - i(\partial_M W^2_N - \partial_N W^2_M + g_{w5}(W^3_M W^1_N - W^1_M W^3_N))] \\
&= \partial_M \frac{W^1_N - iW^2_N}{\sqrt{2}} - \partial_N \frac{W^1_M - iW^2_N}{\sqrt{2}} \\
&\qquad\qquad + \frac{g_{w5}}{\sqrt{2}}\left[-W^3_M\left(W^2_N + iW^1_N\right) + W^3_N\left(W^2_M + iW^1_M\right)\right] \\
&= \partial_M W^+_N - \partial_N W^+_M + \frac{g_{w5}}{\sqrt{2}}\left[-iW^3_M\left(W^1_N - iW^2_N\right) + iW^3_N\left(W^1_M - iW^2_M\right)\right] \\
&= \partial_M W^+_N - \partial_N W^+_M + g_{w5}\left(-iW^3_M W^+_N iW^3_N W^+_M\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\partial_M - i g_{w5} W_M^3\right) W_N^+ - \left(\partial_N - i g_{w5} W_N^3\right) W_M^+ \\
&= \nabla_M W_N^+ - \nabla_N W_M^+
\end{aligned}
$$

where $\nabla_M := \partial_M - i g_{w5} W_M^3$. Similarly,

$$
W_{MN}^- = \nabla_M^\dagger W_N^- - \nabla_N^\dagger W_M^-
$$

We need a term proportional to $\left(W_{MN}^1\right)^2 + \left(W_{MN}^2\right)^2$:

$$
W_{MN}^+ W^{-MN} = \frac{1}{2}\left[\left(W_{MN}^1\right)^2 + \left(W_{MN}^2\right)^2\right]
$$

or

$$
\begin{aligned}
\left(W_{MN}^1\right)^2 + \left(W_{MN}^2\right)^2 &= 2 W_{MN}^+ W^{-MN} \\
&= 2\left(\nabla_M W_N^+ - \nabla_N W_M^+\right)\left(\nabla^{\dagger M} W^{-N} - \nabla^{\dagger N} W^{-M}\right) \\
&= 2(\nabla_M W_N^+ \nabla^{\dagger M} W^{-N} - \nabla_M W_N^+ \nabla^{\dagger N} W^{-M} \\
&\qquad - \underbrace{\nabla_N W_M^+ \nabla^{\dagger M} W^{-N}}_{M \leftrightarrow N} + \underbrace{\nabla_N W_M^+ \nabla^{\dagger N} W^{-M}}_{M \leftrightarrow N}) \\
&= 2(\nabla_M W_N^+ \nabla^{\dagger M} W^{-N} + \nabla_M W_N^+ \nabla^{\dagger M} W^{-N} \\
&\qquad - \nabla_M W_N^+ \nabla^{\dagger N} W^{-M} - \nabla_M W_N^\dagger \nabla^{\dagger N} W^{-M}) \\
&= 4\left(\nabla_M W_N^+ \nabla^{\dagger M} W^{-N} - \nabla_M W_N^+ \nabla^{\dagger N} W^{-M}\right)
\end{aligned}
$$

Now let us take care of the bosons W_M^3 and B_M. In 4D, the basis is rotated by the weak mixing angle, or the so-called Weinberg angle, to produce the physical bosons, Z_M and A_M:

$$
W_M^3 = Z_M c_w + A_M s_w
$$

$$
B_M = -Z_M s_w + A_M c_w
$$

We proceed to evaluate W_{MN}^3 and B_{MN} and see what we get:

$$
\begin{aligned}
W_{MN}^3 &= \partial_M W_N^3 - \partial_N W_M^3 + g_{w5} \varepsilon^{3bc} W_M^b W_N^c \\
&= \partial_M W_N^3 - \partial_N W_M^3 + g_{w5}\left(W_M^1 W_N^2 - W_M^2 W_N^1\right) \\
&= \partial_M \left(Z_N c_w + A_N s_w\right) - \partial_N \left(Z_M c_w + A_M s_w\right) \\
&\qquad + g_{w5}\left(\frac{W_M^+ + W_M^-}{\sqrt{2}} \frac{-W_N^+ + W_N^-}{\sqrt{2} i} - \frac{-W_M^+ + W_M^-}{\sqrt{2} i} \frac{W_N^+ + W_N^-}{\sqrt{2}}\right) \\
&= c_w Z_{MN} + s_w F_{MN} + \frac{g_{w5}}{2i}[-W_M^+ W_N^+ + W_M^+ W_N^- - W_M^- W_N^+ + W_M^- W_N^- \\
&\qquad - (-W_M^+ W_N^+ - W_M^+ W_N^- + W_M^- W_N^+ + W_M^- W_M^-)] \\
&= c_w Z_{MN} + s_w F_{MN} + \frac{g_{w5}}{i}\left(W_M^+ W_N^- - W_M^- W_N^+\right) \\
&= c_w Z_{MN} + s_w F_{MN} - i g_{w5}\left(W_M^+ W_N^- - W_M^- W_N^+\right)
\end{aligned}
$$

where we have defined

$$
Z_{MN} := \partial_M Z_N - \partial_N Z_M
$$

and the good old field strength tensor for the electromagnetic interaction

$$F_{MN} := \partial_M A_N - \partial_N A_M$$

Similarly, for the hypercharge boson, we have

$$\begin{aligned}
B_{MN} &= \partial_M B_N - \partial_N B_M \\
&= \partial_M(-Z_N s_w + A_N c_w) - \partial_N(-Z_M s_w + A_M c_w) \\
&= -s_w Z_{MN} + c_w F_{MN}
\end{aligned}$$

Finally we evaluate the missing components of the kinetic part:

$$\begin{aligned}
\left(W^3_{MN}\right)^2 + \left(B_{MN}\right)^2 &= \left[c_w Z_{MN} + s_w F_{MN} - i g_{w5}\left(W^+_M W^-_N - W^-_M W^+_N\right)\right]^2 \\
&\qquad + \left(-s_w Z_{MN} + c_w F_{MN}\right)^2 \\
&= \left(c_w Z_{MN} + s_w F_{MN}\right)^2 + \left(-s_w Z_{MN} + c_w F_{MN}\right)^2 \\
&\qquad - g^2_{w5}\left(W^+_M W^-_N - W^-_M W^+_N\right)^2 \\
&\qquad - 2 i g_{w5}\left(c_w Z_{MN} + s_w F_{MN}\right)\left(W^+_M W^-_N - W^-_M W^+_N\right) \\
&= \left(Z_{MN}\right)^2 + \left(F_{MN}\right)^2 - 2 g^2_{w5}\left[\left(W^+_M\right)^2\left(W^-_N\right)^2 - \left(W^+_M W^-_N\right)^2\right] \\
&\qquad - 2 i g_{w5}\left(c_w Z_{MN} + s_w F_{MN}\right)\left(W^{+M} W^{-N} - W^{-M} W^{+N}\right)
\end{aligned}$$

Here is a simplification:

$$\begin{aligned}
&\left(c_w Z_{MN} + s_w F_{MN}\right)\left(W^{+M} W^{-N} - W^{-M} W^{+N}\right) \\
&\qquad = \left(c_w Z_{MN} + s_w F_{MN}\right) W^{+M} W^{-N} - \underbrace{\left(c_w Z_{MN} + s_w F_{MN}\right) W^{+N} W^{-M}}_{M \leftrightarrow N} \\
&\qquad = \left(c_w Z_{MN} + s_w F_{MN}\right) W^{+M} W^{-N} - \left(c_w Z_{NM} + s_w F_{NM}\right) W^{+M} W^{-N} \\
&\qquad = \left(c_w Z_{MN} + s_w F_{MN}\right) W^{+M} W^{-N} \\
&\qquad\qquad - \left[-\left(c_w Z_{MN} + s_w F_{MN}\right)\right. \\
&\qquad = 2\left(c_w Z_{MN} + s_w F_{MN}\right) W^{+M} W^{-N}
\end{aligned}$$

At the end of the day, we get

$$\begin{aligned}
\mathscr{L}^{\text{EW}}_{\text{gauge}} &= -\frac{1}{4}\left(W^a_{MN}\right)^2 - \frac{1}{4}\left(B_{MN}\right)^2 \\
&= -\frac{1}{4}\left[\left(W^1_{MN}\right)^2 + \left(W^2_{MN}\right)^2 + \left(W^3_{MN}\right)^2 + \left(B_{MN}\right)^2\right] \\
&= -\frac{1}{4}\{4(\nabla_M W^+_N \nabla^{\dagger M} W^{-N} - \nabla_M W^+_N \nabla^{\dagger N} W^{-M}) + \left(Z_{MN}\right)^2 + \left(F_{MN}\right)^2 \\
&\qquad - 2 g^2_{w5}[\left(W^+_M\right)^2\left(W^-_N\right)^2 - \left(W^+_M W^-_N\right)^2] \\
&\qquad - 2 i g_{w5}\left(c_w Z_{MN} + s_w F_{MN}\right)\left(W^+_M W^-_N - W^-_M W^+_N\right)\} \\
&= \nabla_M W^+_N \nabla^{\dagger M} W^{-N} + \nabla_M W^+_N \nabla^{\dagger N} W^{-M} \\
&\qquad - \frac{1}{4}\left(Z_{MN}\right)^2 - \frac{1}{4}\left(F_{MN}\right)^2 + \frac{1}{2} g^2_{w5}\left[\left(W^+_M\right)^2\left(W^-_N\right)^2 - \left(W^+_M W^-_N\right)^2\right] \\
&\qquad + i g_{w5}\left(c_w Z_{MN} + s_w F_{MN}\right) W^{+M} W^{-N}
\end{aligned}$$

Let us Fourier-expand the fields[10]:

$$V_\mu = \frac{1}{\sqrt{2\pi R}} V_{(0)\mu} + V_{(n)\mu} c_n$$

$$V_5 = V_{(n)5} s_n$$

where $V = \{W^\pm, Z, A\}$. Let us investigate some of the terms that have the potential to produce a mass term in $\mathscr{L}^{\mathrm{EW}}_{\mathrm{gauge}}$:

$$\begin{aligned}
&-\nabla_M W^+_N \nabla^{\dagger M} W^{-N} + \nabla_M W^+_N \nabla^{\dagger N} W^{-M} \\
&\qquad = -\nabla_\mu W^+_\nu \nabla^{+\mu} W^{-\nu} - \nabla_\mu W^+_5 \nabla^{\dagger\mu} W^{-5} \\
&\qquad\quad - \nabla_5 W^+_\mu \nabla^{+5} W^{-\mu} - \nabla_5 W^+_5 \nabla^{\dagger 5} W^{-5} \\
&\qquad\quad + \nabla_\mu W^+_\nu \nabla^{\dagger\nu} W^{-\mu} + \nabla_\mu W^+_5 \nabla^{\dagger 5} W^{-5} \\
&\qquad\quad + \nabla_5 W^+_\mu \nabla^{\dagger\mu} W^{-5} + \nabla_5 W^+_5 \nabla^{\dagger 5} W^{-5} \\
&\qquad = -\nabla_\mu W^+_\nu \nabla^{\dagger\mu} W^{-\nu} + \nabla_\mu W^+_\nu \nabla^{\dagger\nu} W^{-\mu} \\
&\qquad\quad - \nabla_\mu W^+_5 \left(\nabla^{\dagger\mu} W^{-5} - \nabla^{+5} W^{-\mu}\right) \\
&\qquad\quad + \nabla_5 W^+_\mu \left(\nabla^{\dagger\mu} W^{-5} - \nabla^{\dagger 5} W^{-\mu}\right) \\
&\qquad = -\nabla_\mu W^+_\nu \nabla^{\dagger\mu} W^{-\nu} + \nabla_\mu W^+_\nu \nabla^{\dagger\nu} W^{-\mu} \\
&\qquad\quad + \left(\nabla^{\dagger\mu} W^{-5} - \nabla^{+5} W^{-\mu}\right)\left(\nabla_5 W^+_\mu - \nabla_\mu W^+_5\right) \\
&\qquad = -\left[\partial_\mu - i g_{w5}\left(c_w Z_\mu + s_w A_\mu\right)\right] W^+_\nu \left[\partial^\mu + i g_{w5}\left(c_w Z^\mu + s_w A^\mu\right)\right] W^{-\nu} \\
&\qquad\quad + \left[\partial_\mu - i g_{w5}\left[c_w Z_\mu + s_w A_\mu\right]\right] W^+_\nu \left[\partial^\nu + i g_{w5}\left(c_w Z^\nu + s_w A^\nu\right)\right] W^{-\nu} \\
&\qquad\quad + \left\{-\left[\partial^\mu + i g_{w5}\left(c_w Z_\mu + s_w A_\mu\right)\right] W^-_5 + \left[\partial_5 + i g_{w5}\left(c_w Z_5 + s_w A_5\right)\right] W^{-\mu}\right\} \\
&\qquad\quad \times \left\{\left[\partial_5 - i g_{w5}\left(c_w Z_5 + s_w A_5\right)\right] W^+_\mu - \left[\partial_\mu - i g_{w5}\left(c_w Z_\mu + s_w A_\mu\right)\right] W^+_5\right\} \\
&\qquad \supset (\partial_5 W^-_\mu)(\partial_5 W^{+\mu}) - (\partial_\mu W^-_5)(\partial_5 W^{+\mu}) - (\partial_5 W^-_\mu)(\partial^\mu W^+_5) + (\partial_\mu W^-_5)(\partial^\mu W^+_5) \\
&\qquad\quad - (\partial_\mu W^-_\nu)(\partial^\mu W^{+\nu}) + (\partial_\nu W^-_\mu)(\partial^\mu W^{+\mu}) \\
&\qquad\quad + (\text{interaction with } A \text{ and } Z)
\end{aligned}$$

Here, by noting that

$$\partial_5 c_n = -\frac{n}{R} s_n$$

we have

$$\partial_5 W^\pm_\mu = \partial_5 \left(\frac{1}{\sqrt{2\pi R}} W^\pm_{(0)\mu} + W^\pm_{(n)\mu} c_n\right) = -W^\pm_{(n)\mu} \frac{n}{R} s_n$$

$$\partial_\mu W^\pm_5 = \partial_\mu \left(W^\pm_{(n)5} s_n\right) = s_n \partial_\mu W^\pm_{(n)5}$$

$$\partial_\mu W^\pm_\nu = \partial_\mu \left(\frac{1}{\sqrt{2\pi R}} W^\pm_{(0)\nu} + W^\pm_{(n)\nu} c_n\right) = \frac{1}{\sqrt{2\pi R}} \partial_\mu W^\pm_{(0)\nu} + c_n \partial_\mu W^\pm_{(n)\nu}$$

[10] Again, we employ the Einstein summation convention for the KK tower, and we suppress the coordinate dependencies. The fields that carries a KK index are functions of the 4-position only.

so

$$
\begin{aligned}
&-\nabla_M W_N^+ \nabla^{\dagger M} W^{-N} + \nabla_M W_N^+ \nabla^{\dagger N} W^{-M} \\
&\qquad \supset \left(-\frac{n}{R} W_{(n)\mu}^- s_n\right)\left(-\frac{m}{R} W_{(m)}^{+\mu} s_m\right) - \left(s_n \partial_\mu W_{(n)5}^-\right)\left(-\frac{m}{R} W_{(n)}^{+\mu} s_n\right) \\
&\qquad\quad - \left(-\frac{n}{R} W_{(n)\mu}^- s_n\right)\left(s_m \partial^\mu W_{(n)5}^+\right) + \left(s_n \partial_\mu W_{(n)5}^-\right)\left(s_m \partial^\mu W_{(m)5}^+\right) \\
&\qquad\quad - \left(\frac{1}{\sqrt{2\pi R}} \partial_\mu W_{(0)\nu}^- + c_n \partial_\mu W_{(n)\nu}^-\right)\left(\frac{1}{\sqrt{2\pi R}} \partial^\mu W_{(0)}^{+\nu} + c_m \partial^\mu W_{(m)}^{+\nu}\right) \\
&\qquad\quad + \left(\frac{1}{\sqrt{2\pi R}} \partial_\nu W_{(0)\mu}^- + c_n \partial_\nu W_{(n)\mu}^-\right)\left(\frac{1}{\sqrt{2\pi R}} \partial^\mu W_{(0)}^{+\nu} + c_m \partial^\mu W_{(m)}^{+\nu}\right) \\
&\qquad \supset \frac{nm}{R^2} W_{(n)\mu}^- W_{(m)}^{+\mu} s_n s_m + \left(\partial_\mu W_{(n)5}^-\right)\left(\partial^\mu W_{(m)5}^+\right) s_n s_m \\
&\qquad\quad - \left[\frac{1}{2\pi R}\left(\partial_\mu W_{(0)\nu}^-\right)\left(\partial^\mu W_{(0)}^{+\nu}\right) + \left(\partial_\mu W_{(n)\nu}^-\right)\left(\partial^\mu W_{(m)}^{+\nu}\right) c_n c_m\right] \\
&\qquad\quad + \left[\frac{1}{2\pi R}\left(\partial_\nu W_{(0)\mu}^-\right)\left(\partial^\mu W_{(0)}^{+\nu}\right) + \left(\partial_\nu W_{(n)\nu}^-\right)\left(\partial^\mu W_{(m)}^{+\nu}\right) c_n c_m\right] \\
&\qquad \xrightarrow{\int_0^{2\pi R} dy} \frac{n^2}{R^2} W_{(n)\mu}^- W_{(n)}^{+\mu} + \left(\partial_\mu W_{(n)5}^-\right)\left(\partial^\mu W_{(n)5}^+\right) \\
&\qquad\qquad - \left(\partial W_{(0)\nu}^-\right)\left(\partial^\mu W_{(0)}^{+\nu}\right) - \left(\partial_\mu W_{(n)\nu}^-\right)\left(\partial^\mu W_{(n)}^{+\nu}\right) \\
&\qquad\qquad + \left(\partial_\nu W_{(0)\nu}^-\right)\left(\partial^\mu W_{(0)}^{+\nu}\right) + \left(\partial_\nu W_{(n)\mu}^-\right)\left(\partial^\mu W_{(m)}^{+\nu}\right)
\end{aligned}
$$

where we have used the orthogonality integrals of double cosines. The first thing to notice is that we have recovered parts of the kinetic terms for $W_{(0)\mu}^{\pm}$ $W_{(n)\mu}^{\pm}$, and $W_{(n)5}^{\pm}$, and a mass term for $W_{(n)\mu}^{\pm}$:

$$
m_{W,n} = \frac{n}{R}
$$

and $W_5^{\pm}$ seems massless, which we cover again in the Higgs sector. Next, we study a more familiar term in $\mathscr{L}_{\text{gauge}}^{\text{EW}}$:

$$
\begin{aligned}
\left(F_{MN}\right)^2 &= \left(F_{\mu\nu}\right)^2 + \left(F_{\mu 5}\right)^2 + \left(F_{5\mu}\right)^2 + \left(F_{55}\right)^2 \\
&= \left(F_{\mu\nu}\right)^2 + 2\left(F_{\mu 5}\right)^2
\end{aligned}
$$

where we have $F_{55} = 0$ since the field strength tensor is by construction completely antisymmetric, and $F_{5\mu} = -F_{\mu 5}$. Here,

$$
\begin{aligned}
\left(F_{\mu\nu}\right)^2 &= \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 \\
&= 2\left[\left(\partial_\mu A_\nu\right)^2 - \left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\left(F_{\mu 5}\right)^2 &= \left(\partial_\mu A_5 - \partial_5 A_\mu\right)^2 \\
&= \left(\partial_\mu A_5 - \partial_5 A_\mu\right)\left(\partial^\mu A^5 - \partial^5 A^\mu\right) \\
&= \left(\partial_\mu A_5 - \partial_5 A_\mu\right)\left(-\partial^\mu A_5 + \partial_5 A^\mu\right) \\
&= -\left(\partial_\mu A_5\right)^2 - \left(\partial_5 A_\mu\right)^2 + 2\left(\partial_\mu A_5\right)\left(\partial_5 A^\mu\right)
\end{aligned}
$$

where

$$\partial_\mu A_\nu = \partial_\mu \left(\frac{1}{\sqrt{2\pi R}} A_{(0)\nu} + A_{(n)\nu} c_n \right) = \frac{1}{\sqrt{2\pi R}} \partial_\mu A_{(0)\nu} + c_n \partial_\mu A_{(n)\nu}$$

$$\partial_5 A_\mu = \partial_5 \left(\frac{1}{\sqrt{2\pi R}} A_{(0)\mu} + A_{(n)\mu} c_n \right) = -\frac{n}{R} A_{(n)\mu} s_n$$

$$\partial_\mu A_5 = \partial_\mu \left(A_{(n)5} s_n \right) = s_n \partial_\mu A_{(n)5}$$

so

$$\begin{aligned}
(F_{\mu\nu})^2 &= 2\Bigg[\left(\frac{1}{\sqrt{2\pi R}} \partial_\mu A_{(0)\nu} + c_n \partial_\mu A_{(n)\nu} \right) \left(\frac{1}{\sqrt{2\pi R}} \partial^\mu A^\nu_{(0)} + c_m \partial^\mu A^\nu_{(m)} \right) \\
&\qquad - \left(\frac{1}{\sqrt{2\pi R}} \partial_\mu A_{(0)\nu} + c_n \partial_\mu A_{(n)\nu} \right) \left(\frac{1}{\sqrt{2\pi R}} \partial^\nu A^\nu_{(0)} + c_m \partial^\nu A^\mu_{(m)} \right) \Bigg] \\
&= 2\Bigg[\frac{1}{2\pi R} \left(\partial_\mu A_{(0)\nu} \right)^2 + \left(\partial_\mu A_{(n)\nu} \right) \left(\partial^\mu A^\nu_{(m)} \right) c_n c_m \\
&\qquad - \frac{1}{2\pi R} \left(\partial_\mu A_{(0)\nu} \right) \left(\partial^\nu A^\mu_{(0)} \right) - \left(\partial_\mu A_{(n)\nu} \right) \left(\partial^\nu A^\mu_{(m)} \right) c_n c_m \\
&\qquad (\cdots)_n c_n \Bigg] \\
&\xrightarrow{\int_0^{2\pi R} dy} 2\Bigg[\left(\partial_\mu A_{(0)\nu} \right)^2 + \left(\partial_\mu A_{(n)\nu} \right)^2 - \left(\partial_\mu A_{(0)\nu} \right) \left(\partial^\nu A^\mu_{(0)} \right) - \left(\partial_\mu A_{(n)\nu} \right) \left(\partial^\nu A^\mu_{(n)} \right) \Bigg] \\
&= \left(F_{(0)\mu\nu} \right)^2 + \left(F_{(n)\mu\nu} \right)^2
\end{aligned}$$

and similarly

$$\begin{aligned}
(F_{\mu 5})^2 &= -\left(s_n \partial_\mu A_{(n)5} \right) \left(s_m \partial^\mu A_{(m)5} \right) - \left(-\frac{n}{R} A_{(n)\mu} s_n \right) \left(-\frac{m}{R} A^\mu_{(m)} s_m \right) \\
&\qquad + 2 \left(s_n \partial_\mu A_{(n)5} \right) \left(-\frac{m}{R} A^\mu_{(m)} s_n \right) \\
&= -\left(\partial_\mu A_{(n)5} \right) \left(\partial^\mu A_{(m)5} \right) s_n s_m - \frac{nm}{R^2} A_{(n)\mu} A^\mu_{(m)} s_n s_m \\
&\qquad - 2 \frac{m}{R} A^\mu_{(m)} \left(\partial_\mu A_{(n)5} \right) s_n s_m \\
&\xrightarrow{\int_0^{2\pi R} dy} -\left(\partial_\mu A_{(n)5} \right)^2 - \frac{n^2}{R^2} \left(A_{(n)\mu} \right)^2 - \frac{2n}{R} A^\mu_{(n)} \partial_\mu A_{(n)5}
\end{aligned}$$

and totally, we obtain

$$\begin{aligned}
(F_{MN})^2 &\xrightarrow{\int_0^{2\pi R} dy} \left(F_{(0)\mu\nu} \right)^2 + \left(F_{(n)\mu\nu} \right)^2 - 2 \left(\partial_\mu A_{(n)5} \right)^2 - 2 \frac{n^2}{R^2} \left(A_{(n)\mu} \right)^2 \\
&\qquad - \frac{4n}{R} A^\mu_{(n)} \partial_\mu A_{(n)5}
\end{aligned}$$

where a summation is implied over n. The kinetic term for the photon then becomes

$$\mathscr{L}_A \supset -\frac{1}{4}\left(F_{MN}\right)^2 \xrightarrow{\int_0^{2\pi R} dy} -\frac{1}{4}\left(F_{(0)\mu\nu}\right)^2 + \sum_{n\geq 1}\left\{-\frac{1}{4}\left(F_{(n)\mu\nu}\right)^2 + \frac{1}{2}\frac{n^2}{R^2}\left(A_{(n)\mu}\right)^2\right\}$$
$$+\sum_{n\geq 1}\frac{1}{2}\left(\partial_\mu A_{(n)5}\right)^2 + \sum_{n\geq 1}\frac{n}{R}A^\mu_{(n)}\partial_\mu A_{(n)5}$$

Here, we recovered the massless photon in the zeroth mode[11], and obtained a tower of massive KK photons with

$$m_{A,n} = \frac{n}{R}$$

There appears a kinetic term for the scalar $A_{(n)5}$ but it is massless, thus we infer that it is the Goldstone boson for the KK photon. The last term will be canceled out by a gauge-fixing Lagrangian that contains a term like

$$\begin{aligned}
\mathscr{L}_{\text{gf}} &\supset -\frac{1}{2\xi}\left(\partial_\mu A^\mu + \xi\partial_5 A^5\right)^2 \\
&\supset -\frac{1}{2\xi}\left(\partial_\mu A^\mu - \xi\partial_5 A_5\right)^2 \\
&\supset -\frac{1}{2\xi}\left[\left(\partial_\mu A^\mu\right)^2 + \xi^2\left(\partial_5 A_5\right)^2 - 2\xi\left(\partial_\mu A^\mu\right)\left(\partial_5 A_5\right)\right] \\
&\supset -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 - \frac{\xi}{2}\left(\partial_5 A_5\right)^2 + \left(\partial_\mu A^\mu\right)\left(\partial_5 A_5\right)
\end{aligned}$$

Here, ξ is the gauge-fixing parameter of the R_ξ gauge class[12], and

$$\begin{aligned}
\partial_\mu A^\mu &= \frac{1}{\sqrt{2\pi R}}\partial_\mu A^\mu_{(0)} + c_n\partial_\mu A^\mu_{(n)} \\
\partial_5 A_5 &= \frac{n}{R}A_{(n)5}c_n
\end{aligned}$$

so

$$\begin{aligned}
\mathscr{L}_{\text{gf}} \supset& -\frac{1}{2\xi}\left(\frac{1}{\sqrt{2\pi R}}\partial_\mu A^\mu_{(0)} + c_n\partial_\mu A^\mu_{(n)}\right)\left(\frac{1}{\sqrt{2\pi R}}\partial_\nu A^\nu_{(n)} + c_m\partial_\nu A^\nu_{(m)}\right) \\
&- \frac{\xi}{2}\left(\frac{n}{R}A_{(n)5}c_n\right)\left(\frac{m}{R}A_{(m)5}c_m\right) \\
&+ \left(\frac{1}{\sqrt{2\pi R}}\partial_\mu A^\mu_{(0)} + c_n\partial_\mu A^\mu_{(n)}\right)\left(\frac{n}{R}A_{(m)5}c_m\right) \\
\supset& -\frac{1}{2\xi}\left[\frac{1}{2\pi R}\left(\partial_\mu A^\mu_{(0)}\right)^2 + \left(\partial_\mu A^\mu_{(n)}\right)\left(\partial_\nu A^\nu_{(m)}\right)c_n c_m + (\ldots)_n\, c_n\right] \\
&- \frac{\xi}{2}\frac{nm}{R^2}A_{(n)5}A_{(m)5}c_n c_m \; + \frac{n}{R}A_{(m)5}\left(\partial_\mu A^\mu_{(n)}\right)c_n c_m \\
\xrightarrow{\int_0^{2\pi R} dy}& -\frac{1}{2\xi}\left(\partial_\mu A^\mu_{(0)}\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^\mu_{(n)}\right)^2 - \frac{\xi}{2}\frac{n^2}{R^2}\left(A_{(n)5}\right)^2 + \frac{n}{R}A_{(n)5}\partial_\mu A^\mu_{(n)}
\end{aligned}$$

[11] Clearly, this is an illusion. We know from the 4D theory that the actual mass of any gauge boson comes from the Higgs Lagrangian. What we mean above is that there are no new mass terms in the zeroth order, which is not surprising.

[12] At the end, we wish to take $\xi = 1$ to get the Feynman-'t Hooft gauge as desired. Since we can do so virtually at any point in the calculations, we may as well keep it as a parameter.

If we combine the last term here with the one in $\mathscr{L}_A$, we get a total derivative:

$$\begin{aligned}\sum_{n\geq 1}\left(\frac{n}{R}A^{\mu}_{(n)}\partial_\mu A_{(n)5}+\frac{n}{R}A_{(N)5}\partial_\mu A^{\mu}_{(n)}\right) &= \sum_{n\geq 1}\frac{n}{R}\partial_\mu\left(A^{\mu}_{(n)}A_{(n)5}\right)\\ &= \partial_\mu\left[\sum_{n\geq 1}\frac{n}{R}A^{\mu}_{(n)}A_{(n)5}\right]\end{aligned}$$

Since the physics do not change when we add a total derivative of a function that depends on the field variables, we may as well drop this term.

It is beneficial to note that

1. an analysis similar to the one we did to evaluate $(F_{MN})^2$ can be performed on the term $(Z_{MN})^2$, and
2. the rest of the terms in $\mathscr{L}^{\text{EW}}_{\text{gauge}}$ yields interactions and some mixing of fields with 4-gradient of another field, which may be shown to vanish with a suitable gauge fixing in the R_ξ class, such as [Petriolli]

$$\mathscr{L}_{Z,\text{gf}} = -\frac{1}{2\xi}\left(\partial_\mu Z^\mu + \xi\partial_5 Z^5 - \xi m_Z\phi_Z\right)^2$$

$$\mathscr{L}_{W,\text{gf}} = -\frac{1}{\xi}\left(\partial_\mu W^{+\mu} + \xi\partial_5 W^{+5} - \xi i m_W\phi_+\right)\left(\partial_\mu W^{-\mu} + \xi\partial_5 W^{-5} + \xi i m_W\phi_-\right)$$

where m_Z (m_W) is the 4D mass of the Z ($W^\pm$) boson(s) that comes from the kinetic term for the Higgs field.

3.2 The Higgs sector

In the original 4D theory, the masses of the gauge bosons are encoded within the Higgs Lagrangian. The gluons do not interact with the Higgs particle since it does not possess any color charge, so they remain massless. After the spontaneous symmetry breaking, the gauge bosons of the electroweak force, $W^\pm_M$ and Z_M will obtain masses both in the zeroth mode and in the KK levels; the photon will receive a mass only at the KK level. We proceed by evaluating the kinetic term of the Higgs Lagrangian, $|\mathscr{D}_M H|^2$. Later, the potential term will be examined, which turns out to be of less interest.

3.2.1 The kinetic term

The Higgs doublet is given as

$$H = \begin{pmatrix}\phi_+\\ \frac{h+i\phi_Z}{\sqrt{2}}\end{pmatrix}$$

By referring to (2.2), we write

$$\mathscr{D}_{\mu/5}H = \begin{pmatrix}\partial_{\mu/5}\phi_+ - \frac{ig_{w5}}{c_w}\left(\frac{1}{2}-s_w^2\right)Z_{\mu/5}\phi_+ + ie_5A_{\mu/5}\phi_+ - \frac{ig_{w5}}{\sqrt{2}}W^+_{\mu/5}\frac{h+i\phi_Z}{\sqrt{2}}\\ -\frac{ig_{w5}}{\sqrt{2}}W^-_{\mu/5}\phi_+ + \frac{\partial_{\mu/5}h+i\partial_{\mu/5}\phi_Z}{\sqrt{2}} + \frac{ig_{w5}}{c_w}\frac{1}{2}Z_{\mu/5}\frac{h+i\phi_Z}{\sqrt{2}}\end{pmatrix}$$

with

$$(\mathscr{D}_{\mu/5}H)^* = \begin{pmatrix} \partial_{\mu/5}\phi_- + \frac{ig_{w5}}{c_w}\left(\frac{1}{2}-s_w^2\right)Z_{\mu/5}\phi_- - ie_5A_{\mu/5}\phi_- + \frac{ig_{w5}}{\sqrt{2}}W^-_{\mu/5}\frac{h-i\phi_Z}{\sqrt{2}} \\ \frac{ig_{w5}}{\sqrt{2}}W^+_{\mu/5}\phi_- + \frac{\partial_{\mu/5}h-i\partial_{\mu/5}\phi_Z}{\sqrt{2}} - \frac{ig_{w5}}{c_w}\frac{1}{2}Z_{\mu/5}\frac{h-i\phi_Z}{\sqrt{2}} \end{pmatrix}$$

Thus the kinetic part of the Higgs Lagrangian is calculated as

$$\begin{aligned}
\mathscr{L}_{\text{higgs}} &= (\mathscr{D}_\mu H)^\dagger (\mathscr{D}^\mu H) - (\mathscr{D}_5 H)^\dagger (\mathscr{D}_5 H) \\
&\supset -\frac{1}{2}(\partial_5 h)^2 + \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{8}\frac{g_{w5}^2}{c_w^2}h^2(Z_\mu)^2 - \frac{1}{8}\frac{g_{w5}^2}{c_w^2}h^2(Z_5)^2 \\
&\quad + \frac{1}{4}g_{w5}^2h^2W^-_\mu W^{+\mu} - \frac{1}{4}g_{w5}^2h^2W^-_5W^+_5
\end{aligned}$$

Let us Fourier-expand the fields:

$$h = \frac{1}{\sqrt{2\pi R}}(h_{(0)} + v_5) + h_{(n)}c_n$$

$$\begin{aligned}
Z_\mu &= \frac{1}{\sqrt{2\pi R}}Z_{(0)\mu} + Z_{(n)\mu}c_n \\
Z_5 &= Z_{(n)5}s_n
\end{aligned}$$

and similarly for $W^\pm$. The kinetic term for h is read off as

$$\begin{aligned}
\frac{1}{2}(\partial_\mu h)^2 &= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi R}}\partial_\mu h_{(0)} + c_n\partial_\mu h_{(n)}\right)\left(\frac{1}{\sqrt{2\pi R}}\partial^\mu h_{(0)} + c_n\partial^\mu h_{(m)}\right) \\
&= \frac{1}{2}\left[\frac{1}{2\pi R}\left(\partial_\mu h_{(0)}\right)^2 + \left(\partial_\mu h_{(n)}\right)\left(\partial^\mu h_{(m)}\right)c_nc_m + (\cdots)_n c_n\right] \\
&\xrightarrow{\int_0^{2\pi R}dy} \frac{1}{2}\left[\left(\partial_\mu h_{(0)}\right)^2 + \left(\partial_\mu h_{(n)}\right)^2\right]
\end{aligned}$$

Now let us investigate the gauge boson masses. The first thing to notice that the photon remains massless, as expected by construction. For the Z boson, we have

$$\begin{aligned}
\mathscr{L}_{\text{higgs}} &\supset \frac{1}{8}\frac{g_{w5}^2}{c_w^2}h^2(Z_\mu)^2 \\
&\supset \frac{1}{8}\frac{g_{w5}^2}{c_w^2}\left[\frac{1}{\sqrt{2\pi R}}\left(h_{(0)}+v_5\right) + h_{(n)}c_n\right]\left[\frac{1}{\sqrt{2\pi R}}\left(h_{(0)}+v_5\right) + h_{(m)}c_m\right] \\
&\quad \times \left[\frac{1}{\sqrt{2\pi R}}Z_{(0)\mu} + Z_{(k)\mu}c_k\right]\left[\frac{1}{\sqrt{2\pi R}}Z^\mu_{(0)} + Z^\mu_{(l)}c_l\right] \\
&\supset \frac{1}{8}\frac{g_{w5}^2}{c_w^2}\left[\frac{v_5^2}{4\pi^2R^2}\left(Z_{(0)}\mu\right)^2 + \frac{v_5}{2\pi^2R^2}h_{(0)}\left(Z_{(0)\mu}\right)^2 + \frac{1}{4\pi^2R^2}h_{(0)}^2\left(Z_{(0)\mu}\right)^2\right. \\
&\quad + \frac{1}{2\pi R}h_{(m)}h_{(n)}\left(Z_{(0)\mu}\right)^2c_nc_m + \frac{2v_5}{\pi R}h_{(m)}Z_{(0)\mu}Z^\mu_{(k)}c_mc_k \\
&\quad + \frac{2}{\pi R}h_{(0)}h_{(m)}Z_{(0)\mu}Z^\mu_{(k)}c_mc_k + \sqrt{\frac{2}{\pi R}}h_{(n)}h_{(m)}Z_{(0)\mu}Z^\mu_{(k)}c_nc_mc_k \\
&\quad + \frac{v_5^2}{2\pi R}Z_{(k)\mu}Z^\mu_{(l)}c_kc_l + \frac{v_5}{\pi R}h_{(0)}Z_{(k)\mu}Z^\mu_{(l)}c_kc_l + \frac{1}{2\pi R}h_{(0)}^2Z_{(k)\mu}Z^\mu_{(l)}c_kc_l
\end{aligned}$$

$$v_5\sqrt{\frac{2}{\pi R}}h_{(m)}Z_{(k)\mu}Z^{\mu}_{(l)}c_m c_k c_l + \sqrt{\frac{2}{\pi R}}h_{(0)}h_{(m)}Z_{(k)\mu}Z^{\mu}_{(l)}c_m c_k c_l$$

$$\left. + h_{(m)}h_{(n)}Z_{(k)\mu}Z^{\mu}_{(l)}c_n c_m c_k c_l + (\cdots)_n\, c_n \right]$$

$$\xrightarrow{\int_0^{2\pi R} dy} \frac{1}{8}\frac{g_{w5}^2}{c_w^2}\left[\frac{v_5^2}{2\pi R}\left(Z_{(0)}\mu\right)^2 + \frac{v_5}{\pi R}h_{(0)}\left(Z_{(0)\mu}\right)^2 + \frac{1}{2\pi R}h_{(0)}^2\left(Z_{(0)\mu}\right)^2\right.$$

$$+ \frac{1}{2\pi R}h_{(n)}h_{(n)}\left(Z_{(0)\mu}\right)^2 + \frac{2v_5}{\pi R}h_{(n)}Z_{(0)\mu}Z^{\mu}_{(n)}$$

$$+ \frac{2}{\pi R}h_{(0)}h_{(n)}Z_{(0)\mu}Z^{\mu}_{(n)} + \sqrt{\frac{2}{\pi R}}h_{(n)}h_{(m)}Z_{(0)\mu}Z^{\mu}_{(k)}\Delta_{nmk}$$

$$+ \frac{v_5^2}{2\pi R}Z_{(n)\mu}Z^{\mu}_{(n)} + \frac{v_5}{\pi R}h_{(0)}Z_{(n)\mu}Z^{\mu}_{(n)} + \frac{1}{2\pi R}h_{(0)}^2 Z_{(n)\mu}Z^{\mu}_{(n)}$$

$$v_5\sqrt{\frac{2}{\pi R}}h_{(m)}Z_{(k)\mu}Z^{\mu}_{(l)}\Delta_{mkl} + \sqrt{\frac{2}{\pi R}}h_{(0)}h_{(m)}Z_{(k)\mu}Z^{\mu}_{(l)}\Delta_{mkl}$$

$$\left. + h_{(m)}h_{(n)}Z_{(k)\mu}Z^{\mu}_{(l)}\Delta_{nmkl}\right]$$

The mass of the Z boson at the zero level – in the real world – is found via the comparison

$$\frac{1}{2}m_Z^2\left(Z_{(0)\mu}\right)^2 = \frac{1}{8}\frac{g_{w5}^2}{c_w^2}\frac{v_5^2}{2\pi R}\left(Z_{(0)\mu}\right)^2$$

so

$$m_Z^2 = \frac{g_{w5}v_5}{2\sqrt{2\pi R}c_w}$$

This relation alone signals that the Higgs VEV in 5D is related to its counterpart in 4D[13] through

$$v = \frac{v_5}{\sqrt{2\pi R}}$$

By expanding and reducing (integrating out) the gauge Lagrangian, we have found that

$$m_{Z(n)}^2 = \frac{n^2}{R^2}$$

These add up to produce the effective mass

$$M_Z := \sqrt{m_{Z(n)}^2 + m_Z^2}$$

The analysis for the $W^{\pm}$ bosons are the same, provided that we send one of the Z's in the products to W^- and the other to W^+, and that the overall coefficient now becomes $\frac{1}{4}g_{w5}^2$ instead of $\frac{1}{8}\frac{g_{w5}^2}{c_w^2}$.

[13] We will observe this also in the next section when we find that $\mu = \mu_5$ and $\lambda = \frac{\lambda_5}{2\pi R}$. Since $v = \sqrt{\mu^2/\lambda}$, and similarly in 5D, we recover the mentioned relation.

Let us examine one more term:

$$
\begin{aligned}
-\frac{1}{2}\left(\partial_5 h\right)^2 &= -\frac{1}{2}\left[\partial_5\left(\frac{1}{\sqrt{2\pi R}}h_{(0)} + h_{(n)}c_n\right)\right]\left[\partial_5\left(\frac{1}{\sqrt{2\pi R}}h_{(0)} + h_{(m)}c_m\right)\right] \\
&= -\frac{1}{2}\left(h_{(n)}\partial_5 c_n\right)\left(h_{(m)}\partial_5 c_m\right) \\
&= -\frac{1}{2}h_{(n)}h_{(m)}\frac{nm}{R^2}s_n s_m \\
&\xrightarrow{\int_0^{2\pi R} dy} -\frac{1}{2}\frac{n^2}{R^2}\left(h_{(n)}\right)^2
\end{aligned}
$$

We also have a kinetic term for $h_{(n)}$ from the expansion of $\left(\partial_\mu h\right)^2$, so we combine them:

$$
\mathscr{L}_{\text{higgs}} \xrightarrow{\int_0^{2\pi R} dy} \cdots \supset \sum_{n\geq 1}\frac{1}{2}\left(\partial_\mu h_{(n)}\right)^2 - \frac{1}{2}\frac{n^2}{R^2}h_{(n)}^2
$$

Hence we get a 5D Higgs here with mass n/R.

There remains the charged and neutral scalars, $\phi_\pm$ and ϕ_Z. The best practice is to combine them with $W_5^\pm$ and Z_5 to form new scalar. At this point, we should take into consideration the full expansion of the kinetic term of the Higgs field, which we produce using Mathematica[14] (Figure 2).

[14]As model-builders, we want to implement our model piece by piece on Mathematica. Of course, expanding a product is not a big deal unless we also expand the fields in Fourier series. This will be done once we are able to talk to the software that, for instance, the index summations $\mathcal{K}_{1(n)}\mathcal{K}_{2(m)}c_n c_m$ and $\mathcal{K}_{1(k)}\mathcal{K}_{2(l)}c_k c_l$ refer to the same quantity.

$$
\begin{aligned}
&-\frac{\mathrm{d5h}^2}{2}+\frac{\mathrm{dh}^2}{2}-\mathrm{d5}\phi_m\,\mathrm{d5}\phi_p-\frac{\mathrm{d5}\phi_z^2}{2}+\mathrm{d}\phi_m\,\mathrm{d}\phi_p+\frac{\mathrm{d}\phi_z^2}{2}-\frac{h\,Z5\,\mathrm{d5}\phi_z\,g_{w5}}{2\,c_w}+\frac{h\,Z\,\mathrm{d}\phi_z\,g_{w5}}{2\,c_w}+\frac{h^2\,Z^2\,g_{w5}^2}{8\,c_w^2}-\frac{h^2\,Z5^2\,g_{w5}^2}{8\,c_w^2}+\\
&\frac{1}{2}\,i\,h\,\mathrm{d}\phi_p\,g_{w5}\,W_m-\frac{1}{2}\,i\,h\,\mathrm{d}\phi_m\,g_{w5}\,W_p+\frac{1}{4}\,h^2\,g_{w5}^2\,W_m\,W_p-\frac{1}{2}\,i\,h\,\mathrm{d5}\phi_p\,g_{w5}\,W5_m+\frac{1}{2}\,i\,h\,\mathrm{d5}\phi_m\,g_{w5}\,W5_p-\\
&\frac{1}{4}\,h^2\,g_{w5}^2\,W5_m\,W5_p+i\,A5\,\mathrm{d5}\phi_p\,e_5\,\phi_m-i\,A\,\mathrm{d}\phi_p\,e_5\,\phi_m-\frac{i\,Z5\,\mathrm{d5}\phi_p\,g_{w5}\,\phi_m}{2\,c_w}+\frac{i\,Z\,\mathrm{d}\phi_p\,g_{w5}\,\phi_m}{2\,c_w}+\\
&\frac{i\,Z5\,\mathrm{d5}\phi_p\,g_{w5}\,s_w^2\,\phi_m}{c_w}-\frac{i\,Z\,\mathrm{d}\phi_p\,g_{w5}\,s_w^2\,\phi_m}{c_w}+\frac{1}{2}\,i\,\mathrm{dh}\,g_{w5}\,W_p\,\phi_m-\frac{1}{2}\,\mathrm{d}\phi_z\,g_{w5}\,W_p\,\phi_m-\frac{1}{2}\,A\,h\,e_5\,g_{w5}\,W_p\,\phi_m-\\
&\frac{h\,Z\,g_{w5}^2\,s_w^2\,W_p\,\phi_m}{2\,c_w}-\frac{1}{2}\,i\,\mathrm{d5h}\,g_{w5}\,W5_p\,\phi_m+\frac{1}{2}\,\mathrm{d5}\phi_z\,g_{w5}\,W5_p\,\phi_m+\frac{1}{2}\,A5\,h\,e_5\,g_{w5}\,W5_p\,\phi_m+\frac{h\,Z5\,g_{w5}^2\,s_w^2\,W5_p\,\phi_m}{2\,c_w}-\\
&i\,A5\,\mathrm{d5}\phi_m\,e_5\,\phi_p+i\,A\,\mathrm{d}\phi_m\,e_5\,\phi_p+\frac{i\,Z5\,\mathrm{d5}\phi_m\,g_{w5}\,\phi_p}{2\,c_w}-\frac{i\,Z\,\mathrm{d}\phi_m\,g_{w5}\,\phi_p}{2\,c_w}-\frac{i\,Z5\,\mathrm{d5}\phi_m\,g_{w5}\,s_w^2\,\phi_p}{c_w}+\\
&\frac{i\,Z\,\mathrm{d}\phi_m\,g_{w5}\,s_w^2\,\phi_p}{c_w}-\frac{1}{2}\,i\,\mathrm{dh}\,g_{w5}\,W_m\,\phi_p-\frac{1}{2}\,\mathrm{d}\phi_z\,g_{w5}\,W_m\,\phi_p-\frac{1}{2}\,A\,h\,e_5\,g_{w5}\,W_m\,\phi_p-\frac{h\,Z\,g_{w5}^2\,s_w^2\,W_m\,\phi_p}{2\,c_w}+\\
&\frac{1}{2}\,i\,\mathrm{d5h}\,g_{w5}\,W5_m\,\phi_p+\frac{1}{2}\,\mathrm{d5}\phi_z\,g_{w5}\,W5_m\,\phi_p+\frac{1}{2}\,A5\,h\,e_5\,g_{w5}\,W5_m\,\phi_p+\frac{h\,Z5\,g_{w5}^2\,s_w^2\,W5_m\,\phi_p}{2\,c_w}+A^2\,e_5^2\,\phi_m\,\phi_p-\\
&A5^2\,e_5^2\,\phi_m\,\phi_p-\frac{A\,Z\,e_5\,g_{w5}\,\phi_m\,\phi_p}{c_w}+\frac{A5\,Z5\,e_5\,g_{w5}\,\phi_m\,\phi_p}{c_w}+\frac{Z^2\,g_{w5}^2\,\phi_m\,\phi_p}{4\,c_w^2}-\frac{Z5^2\,g_{w5}^2\,\phi_m\,\phi_p}{4\,c_w^2}+\frac{2\,A\,Z\,e_5\,g_{w5}\,s_w^2\,\phi_m\,\phi_p}{c_w}-\\
&\frac{2\,A5\,Z5\,e_5\,g_{w5}\,s_w^2\,\phi_m\,\phi_p}{c_w}-\frac{Z^2\,g_{w5}^2\,s_w^2\,\phi_m\,\phi_p}{c_w^2}+\frac{Z5^2\,g_{w5}^2\,s_w^2\,\phi_m\,\phi_p}{c_w^2}+\frac{Z^2\,g_{w5}^2\,s_w^4\,\phi_m\,\phi_p}{c_w^2}-\frac{Z5^2\,g_{w5}^2\,s_w^4\,\phi_m\,\phi_p}{c_w^2}+\\
&\frac{1}{2}\,g_{w5}^2\,W_m\,W_p\,\phi_m\,\phi_p-\frac{1}{2}\,g_{w5}^2\,W5_m\,W5_p\,\phi_m\,\phi_p-\frac{\mathrm{dh}\,Z\,g_{w5}\,\phi_z}{2\,c_w}+\frac{\mathrm{d5h}\,Z5\,g_{w5}\,\phi_z}{2\,c_w}+\frac{1}{2}\,\mathrm{d}\phi_p\,g_{w5}\,W_m\,\phi_z+\\
&\frac{1}{2}\,\mathrm{d}\phi_m\,g_{w5}\,W_p\,\phi_z-\frac{1}{2}\,\mathrm{d5}\phi_p\,g_{w5}\,W5_m\,\phi_z-\frac{1}{2}\,\mathrm{d5}\phi_m\,g_{w5}\,W5_p\,\phi_z-\frac{1}{2}\,i\,A\,e_5\,g_{w5}\,W_p\,\phi_m\,\phi_z-\frac{i\,Z\,g_{w5}^2\,s_w^2\,W_p\,\phi_m\,\phi_z}{2\,c_w}+\\
&\frac{1}{2}\,i\,A5\,e_5\,g_{w5}\,W5_p\,\phi_m\,\phi_z+\frac{i\,Z5\,g_{w5}^2\,s_w^2\,W5_p\,\phi_m\,\phi_z}{2\,c_w}+\frac{1}{2}\,i\,A\,e_5\,g_{w5}\,W_m\,\phi_p\,\phi_z+\frac{i\,Z\,g_{w5}^2\,s_w^2\,W_m\,\phi_p\,\phi_z}{2\,c_w}-\\
&\frac{1}{2}\,i\,A5\,e_5\,g_{w5}\,W5_m\,\phi_p\,\phi_z-\frac{i\,Z5\,g_{w5}^2\,s_w^2\,W5_m\,\phi_p\,\phi_z}{2\,c_w}+\frac{Z^2\,g_{w5}^2\,\phi_z^2}{8\,c_w^2}-\frac{Z5^2\,g_{w5}^2\,\phi_z^2}{8\,c_w^2}+\frac{1}{4}\,g_{w5}^2\,W_m\,W_p\,\phi_z^2-\frac{1}{4}\,g_{w5}^2\,W5_m\,W5_p\,\phi_z^2
\end{aligned}
$$

Figure 2: MATHEMATICA output of the kinetic term of the Higgs doublet, $|\mathscr{D}_M H|^2$. In this notation, for example, `dh` and `d5h` refer to $\partial_\mu h$ and $\partial_5 h$, respectively. The usual Lorentz indices of the vector fields have been omitted. The subscript `m` and `p` denote $-$ and $+$ sign, respectively. The notation for the fifth components of the vector fields is a `5` tacked in to the field name.

But let us take one step ahead. Unless we take the derivative of a Z_2-odd field – the fifth component of all the vector fields and some left (right) component of a right- (left-)chiral fermion – we see that

$$\partial_5 \mathcal{K} = -\frac{n}{R}\mathcal{K}_{(n)} s_n$$

and if this term is squared, it will necessarily give a mass term for the KK field of interest.

Meantime, we define these new scalars as [21]

$$\chi^0_{(n)} := \frac{(n/R)\phi_{Z(n)} - m_Z Z_{(n)5}}{\sqrt{m_Z^2 + \frac{n^2}{R^2}}}$$

$$\chi^{\pm}_{(n)} := \frac{(n/R)\phi_{\pm(n)} \pm i m_W W^{\pm}_{(n)5}}{\sqrt{m_W^2 + \frac{n^2}{R^2}}}$$

where m_Z (m_W) is the mass of $Z_{(0)}$ ($W^{\pm}_{(0)}$). In the literature, $\chi^0_{(n)}$ is an electrically neutral, physical, CP-odd scalar, whereas the $\chi^{\pm}_{(n)}$ are the charged Higgs scalars. Other notations involve $\{H^0_{(n)}, H^{\pm}_{(n)}\}$ and $\{a^0_{(n)}, a^{\pm}_{(n)}\}$.

Without getting lost in the jungle of these expansions, it is notoriously difficult to pick out the χ scalars. We regret to save this exercise for a later work.

3.2.2 The potential term

The potential term in the Higgs Lagrangian reads

$$\begin{aligned}
\mathscr{U}_{\text{higgs}} &= \mu_5^2 H^\dagger H - \lambda_5 \left(H^\dagger H\right)^2 \\
&= \mu_5^2 \begin{pmatrix} \phi_- & \frac{h - i\phi_Z}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \frac{h + i\phi_Z}{\sqrt{2}} \end{pmatrix} - \lambda_5 \left[\begin{pmatrix} \phi_- & \frac{h - i\phi_Z}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \frac{h + i\phi_Z}{\sqrt{2}} \end{pmatrix} \right]^2
\end{aligned}$$

Here,

$$\begin{aligned}
H^\dagger H &= \left(\frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} + H^\dagger_{(n)} c_n \right) \left(\frac{1}{\sqrt{2\pi R}} H_{(0)} + H_{(m)} c_m \right) \\
&= \frac{1}{2\pi R} H^\dagger_{(0)} H_{(0)} + H^\dagger_{(n)} H_{(m)} c_n c_m + (\cdots)_n \, c_n \\
&\xrightarrow{\int_0^{2\pi R} dy} H^\dagger_{(0)} H_{(0)} + H^\dagger_{(n)} H_{(n)}
\end{aligned}$$

and

$$\begin{aligned}
\left(H^\dagger H\right)^2 &= \left[\left(\frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} + H^\dagger_{(n)} c_n \right) \left(\frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} + H^\dagger_{(m)} c_m \right) \right] \\
&\quad \times \left[\left(\frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} + H^\dagger_{(k)} c_k \right) \left(\frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} + H^\dagger_{(l)} c_l \right) \right] \\
&= \left(\frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} H_{(0)} + H^\dagger_{(n)} H_{(m)} c_n c_m + \frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} H_{(n)} c_n + \frac{1}{\sqrt{2\pi R}} H^\dagger_{(n)} H_{(0)} c_n \right) \\
&\quad \times \left(\frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} H_{(0)} + H^\dagger_{(k)} H_{(l)} c_k c_l + \frac{1}{\sqrt{2\pi R}} H^\dagger_{(0)} H_{(k)} c_k + \frac{1}{\sqrt{2\pi R}} H^\dagger_{(k)} H_{(0)} c_k \right) \\
&= \frac{1}{4\pi^2 R^2} \left(H^\dagger_{(0)} H_{(0)} \right) \left(H^\dagger_{(0)} H_{(0)} \right) + \frac{1}{\pi R} \left(H^\dagger_{(0)} H_{(0)} \right) \left(H^\dagger_{(n)} H_{(m)} \right) c_n c_m \\
&\quad + \left(H^\dagger_{(n)} H_{(m)} \right) \left(H^\dagger_{(k)} H_{(l)} \right) c_n c_m c_k c_l \\
&\quad + \sqrt{\frac{2}{\pi R}} \left(H^\dagger_{(n)} H_{(m)} \right) \left(H^\dagger_{(0)} H_{(k)} \right) c_n c_m c_k \\
&\quad + \sqrt{\frac{2}{\pi R}} \left(H^\dagger_{(n)} H_{(m)} \right) \left(H^\dagger_{(k)} H_{(0)} \right) c_n c_m c_k \\
&\quad + \frac{1}{2\pi R} \left(H^\dagger_{(0)} H_{(n)} \right) \left(H^\dagger_{(0)} H_{(m)} \right) c_n c_m \\
&\quad + \frac{1}{\pi R} \left(H^\dagger_{(0)} H_{(n)} \right) \left(H^\dagger_{(m)} H_{(0)} \right) c_n c_m
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2\pi R}\left(H^\dagger_{(n)}H_{(0)}\right)\left(H^\dagger_{(m)}H_{(0)}\right)c_n c_m \\
\xrightarrow{\int_0^{2\pi R}dy}\ & \frac{1}{2\pi R}\left(H^\dagger_{(0)}H_{(0)}\right)^2 + \frac{1}{\pi R}\left(H^\dagger_{(0)}H_{(0)}\right)\left(H^\dagger_{(n)}H_{(n)}\right) \\
&+ \left(H^\dagger_{(n)}H_{(m)}\right)\left(H^\dagger_{(k)}H_{(l)}\right)\Delta_{nmkl} \\
&+ \sqrt{\frac{2}{\pi R}}\left(H^\dagger_{(n)}H_{(m)}\right)\left(H^\dagger_{(0)}H_{(k)}\right)\Delta_{nmk} \\
&+ \sqrt{\frac{2}{\pi R}}\left(H^\dagger_{(n)}H_{(m)}\right)\left(H^\dagger_{(k)}H_{(0)}\right)\Delta_{nmk} \\
&+ \frac{1}{2\pi R}\left(H^\dagger_{(0)}H_{(n)}\right)\left(H^\dagger_{(0)}H_{(n)}\right) \\
&+ \frac{1}{\pi R}\left(H^\dagger_{(0)}H_{(n)}\right)\left(H^\dagger_{(n)}H_{(0)}\right) \\
&+ \frac{1}{2\pi R}\left(H^\dagger_{(n)}H_{(0)}\right)\left(H^\dagger_{(n)}H_{(0)}\right)
\end{aligned}
$$

When we tack in the factors μ_5 and λ_5 to $H^\dagger H$ and to $\left(H^\dagger H\right)^2$, respectively, we see that

$$\mu = \mu_5$$

$$\lambda = \frac{\lambda_5}{2\pi R}$$

The Higgs self-interactions can be read here directly – we just omit the daggers and let $H_{(j)} \to h_{(j)}/\sqrt{2}$. The interactions with and among $\phi_\pm$ and ϕ_Z would be more interesting if we could find a short cut to replace the parameters on the expansions with the new scalars χ^0 and $\chi^\pm$.

3.3 The fermion sector

Let us recall the fermion Lagrangian:

$$\mathscr{L}_{\text{fermion}} = \bar{q}i\Gamma^M \mathscr{D}^q_M q + \bar{\mathcal{Q}}i\Gamma^M \mathscr{D}^{\mathcal{Q}}_M \mathcal{Q} + \bar{u}i\Gamma^M \mathscr{D}^u_M u + \bar{d}i\Gamma^M \mathscr{D}^d_M + \bar{L}i\Gamma^M \mathscr{D}^L_M L + \bar{l}i\Gamma^M \mathscr{D}^l_M l$$

In this section, we will examine the singlets and doublets, leaving the pure quark-gluon interaction to its fate. But first, we will summarize some identities that involves the Dirac matrices.

3.3.1 A summary of useful identities

The projection operators have been formed by using the fifth Dirac matrix:

$$P_{R/L} := \frac{1 \pm \gamma^5}{2}$$

First of all, by definition, we have

$$P_R + P_L = 1$$

Since $\{\gamma^\mu, \gamma^5\} = 0$ where the curly brackets define the anti-commutator, we infer that

$$P_L\gamma^\mu = \gamma^\mu P_R$$

Furthermore, we have $\left(\gamma^5\right)^2 = 1$ and $\left(\gamma^5\right)^\dagger$. The former implies

$$P_R^2 = P_R, \qquad P_L^2 = P_L, \qquad P_R\gamma^5 = P_R, \qquad P_L\gamma^5 = -P_L$$

and the latter helps give

$$P_{R/L}^\dagger = P_{R/L}$$

Finally, in a Dirac current, there will always appear an adjoint field, $\bar{\psi}$, and if it carries a chirality, then we will have to save it from the bar:

$$\overline{P_{R/L}\psi} = \left(P_{R/L}\psi\right)^\dagger \gamma^0 = \psi^\dagger P_{R/L}^\dagger \gamma^0 = \psi^\dagger P_{R/L}\gamma^0 = \psi^\dagger\gamma^0 P_{L/R} = \bar{\psi}P_{L/R}$$

by using $\{\gamma^0, \gamma^5\} = 0$.

3.3.2 The lepton singlet

Here, we will analyze the lepton singlet. The results for the quark singlet can be obtained by letting $Z_M \to \frac{1}{3}Z_M$ and $A_M \to \frac{1}{3}A_M$, which is clear when we compare (2.3) and (2.4). We proceed with evaluating the operator, $\Gamma^M \mathscr{D}_M^l$:

$$\begin{aligned}
\Gamma^M \mathscr{D}_M^l &= \Gamma^\mu \mathscr{D}_\mu^l + \Gamma^5 \mathscr{D}_5^l \\
&= \gamma^\mu \mathscr{D}_\mu^l + i\gamma^5 \mathscr{D}_5^l \\
&= \gamma^\mu \left(\partial_\mu - i\frac{g_{w5}}{c_w} s_w^2 Z_\mu - ie_5 A_\mu\right) + i\gamma^5 \left(\partial_5 - i\frac{g_{w5}}{c_w} s_w^2 Z_5 - ie_5 A_5\right) \\
&= \gamma^\mu\partial^\mu - i\frac{g_{w5}}{c_w} s_w^2 \gamma^\mu Z_\mu - ie_5\gamma^\mu A_\mu + i\gamma^5\partial_5 + \frac{g_{w5}}{c_w} s_w^2 \gamma^5 Z_5 + e_5\gamma^5 A_5 \\
&= \gamma^\mu\partial_\mu - i\frac{g_{w5}}{c_w} s_w^2 \gamma^\mu \left(\frac{1}{\sqrt{2\pi R}} Z_{(0)\mu} + Z_{(k)\mu}c_k\right) - ie_5\gamma^\mu \left(\frac{1}{\sqrt{2\pi R}} A_{(0)\mu} + A_{(k)\mu}c_k\right) \\
&\quad + i\gamma^5\partial_5 + \frac{g_{w5}}{c_w} s_w^2 \gamma^5 Z_{(k)5}s_k + e_5\gamma^5 A_{(k)5}s_k
\end{aligned}$$

Let us hit this on the singlet field:

$$\begin{aligned}
\Gamma^M \mathscr{D}_M^l l = &\left[\gamma^\mu\partial_\mu - i\frac{g_{w5}}{c_w} s_w^2 \gamma^\mu \left(\frac{1}{\sqrt{2\pi R}} Z_{(0)\mu} + Z_{(k)\mu}c_k\right) - ie_5\gamma^\mu \left(\frac{1}{\sqrt{2\pi R}} A_{(0)\mu} + A_{(k)\mu}c_k\right)\right. \\
&\quad \left. + i\gamma^5\partial_5 + \frac{g_{w5}}{c_w} s_w^2 \gamma^5 Z_{(k)5}s_k + e_5\gamma^5 A_{(k)5}s_k\right] \left(\frac{1}{\sqrt{2\pi R}} l_{(0)} + P_R l_{(n)}c_n + P_L l_{(n)}s_n\right) \\
= &\frac{1}{\sqrt{2\pi R}}\gamma^\mu \partial\mu l_{(0)} + \gamma^\mu P_R c_n \partial_\mu l_{(n)} + \gamma^\mu P_L s_n \partial_\mu l_{(n)} \\
&- \frac{1}{2\pi R} i\frac{g_{w5}}{c_w} s_w^2 Z_{(0)\mu} l_{(0)} - i\frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu Z_{(k)\mu} c_k l_{(0)} \\
&- i\frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu P_R Z_{(0)\mu} c_n l_{(n)} - i\frac{g_{w5}}{c_w} s_w^2 \gamma^\mu P_R Z_{(k)\mu} c_k c_n l_{(n)} \\
&- i\frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu P_L Z_{(0)\mu} s_n l_{(n)} - i\frac{g_{w5}}{c_w} s_w^2 \gamma^\mu P_L Z_{(k)\mu} c_k s_n l_{(n)} \\
&- \frac{1}{2\pi R} ie_5\gamma^\mu A_{(0)\mu} l_{(0)} - i\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu A_{(k)\mu} c_k l_{(0)}
\end{aligned}$$

$$
\begin{aligned}
&- i\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_R A_{(0)\mu} c_n l_{(n)} - ie_5 \gamma^\mu P_R A_{(k)\mu} c_k c_n l_{(n)} \\
&- i\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_L A_{(0)\mu} s_n l_{(n)} - ie_5 \gamma^\mu P_L A_{(k)\mu} c_k s_n l_{(n)} \\
&- i\gamma^5 P_R \frac{n}{R} s_n l_{(n)} + i\gamma^5 P_L \frac{n}{R} c_n l_{(n)} \\
&+ \frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^5 Z_{(k)5} s_k l_{(0)} + \frac{g_{w5}}{c_w} s_w^2 \gamma^5 P_R Z_{(k)5} c_n l_{(n)} \\
&+ \frac{g_{w5}}{c_w} s_w^2 \gamma^5 P_L Z_{(k)5} s_k s_n l_{(n)} + \frac{e_5}{\sqrt{2\pi R}}\gamma^5 A_{(k)5} s_k l_{(0)} \\
&+ e_5 \gamma^5 P_R A_{(k)5} s_k c_n l_{(n)} + e_5 \gamma^5 P_L A_{(k)5} s_k s_n l_{(n)}
\end{aligned}
$$

Finally, let us hit the adjoint singlet on these terms:

$$
\begin{aligned}
\bar{l} i\Gamma^M \mathscr{D}^l_M l &= \left(\frac{1}{\sqrt{2\pi R}}\bar{l}_{(0)} + \overline{P_R l_{(m)}} c_m + \overline{P_L l_{(m)}} s_m\right) i\Gamma^M \mathscr{D}^l_M l \\
&= \left(\frac{1}{\sqrt{2\pi R}}\bar{l}_{(0)} + \bar{l}_{(m)} P_L c_m + \bar{l}_{(m)} P_R s_m\right) i\Gamma^M \mathscr{D}^l_M l \\
&\supset \frac{1}{2\pi R}\bar{l}_{(0)} i\gamma^\mu \partial_\mu l_{(0)} + \frac{1}{2\pi R}\bar{l}_{(0)} \frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu Z_{(0)\mu} l_{(0)} \\
&\quad + \bar{l}_{(0)} \frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu P_R Z_{(k)\mu} c_k c_n l_{(n)} + \frac{1}{2\pi R}\bar{l}_{(0)} \frac{e_5}{\sqrt{2\pi R}}\gamma^\mu A_{(0)\mu} l_{(0)} \\
&\quad + \bar{l}_{(0)} \frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_R A_{(k)\mu} c_k c_n l_{(n)} + \bar{l}_{(0)} i\frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^5 P_L Z_{(k)5} s_k s_n l_{(n)} \\
&\quad + \bar{l}_{(0)} i\frac{e_5}{\sqrt{2\pi R}}\gamma^5 P_L A_{(k)5} s_k s_n l_{(n)} + \bar{l}_{(m)} i\gamma^\mu P_R c_m c_n \partial_\mu l_{(n)} \\
&\quad + \bar{l}_{(m)} \frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu P_R Z_{(0)\mu} c_m c_n l_{(n)} + \bar{l}_{(m)} \frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_R A_{(k)\mu} c_m c_k l_{(0)} \\
&\quad + \bar{l}_{(m)} \frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_R A_{(0)\mu} c_m c_n l_{(n)} + \bar{l}_{(m)} e_5 \gamma^\mu P_R A_{(k)\mu} c_m c_k c_n l_{(n)} \\
&\quad + \bar{l}_{(m)} P_L \frac{n}{R} c_n c_m l_{(n)} - \bar{l}_{(m)} i\frac{g_{w5}}{c_w} s_w^2 P_L Z_{(k)5} s_k s_n c_m l_{(n)} \\
&\quad - \bar{l}_{(m)} ie_5 P_L A_{(k)5} s_k s_n c_m l_{(n)} + \bar{l}_{(m)} i\gamma^\mu P_L s_m s_n \partial_\mu l_{(n)} \\
&\quad + \bar{l}_{(m)} \frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 P_L Z_{(0)\mu} s_m s_n l_{(n)} + \bar{l}_{(m)} \frac{g_{w5}}{c_w} s_w^2 \gamma^\mu P_L Z_{(k)\mu} s_m s_n c_k l_{(n)} \\
&\quad + \bar{l}_{(m)} \frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_L A_{(0)\mu} s_m s_n l_{(n)} + \bar{l}_{(m)} e_5 \gamma^\mu P_L A_{(k)\mu} s_n s_m c_k l_{(n)} \\
&\quad + \bar{l}_{(m)} P_R \frac{n}{R} s_n s_m l_{(n)} + \bar{l}_{(m)} i\frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 P_R Z_{(k)5} s_m s_k l_{(0)} \\
&\quad + \bar{l}_{(m)} i\frac{g_{w5}}{c_w} s_w^2 P_R Z_{(k)5} s_m s_k c_n l_{(n)} + \bar{l}_{(m)} i\frac{e_5}{\sqrt{2\pi R}} P_R A_{(k)5} s_m s_k l_{(0)} \\
&\quad + \bar{l}_{(m)} ie_5 P_R A_{(k)5} s_m s_k c_n l_{(n)} + \text{terms that give zero when integrated} \\
&\xrightarrow{\int_0^{2\pi R} dy} \bar{l}_{(0)} i\gamma^\mu \partial_\mu l_{(0)} + \bar{l}_{(0)} \frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu Z_{(0)\mu} l_{(0)} \\
&\quad + \bar{l}_{(0)} \frac{g_{w5}/\sqrt{2\pi R}}{c_w} s_w^2 \gamma^\mu P_R Z_{(n)\mu} l_{(n)} + \bar{l}_{(0)} \frac{e_5}{\sqrt{2\pi R}}\gamma^\mu A_{(0)\mu} l_{(0)}
\end{aligned}
$$

$$
\begin{aligned}
&+\bar{l}_{(0)}\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_R A_{(n)\mu}l_{(n)}+\bar{l}_{(0)}i\frac{g_{w5}/\sqrt{2\pi R}}{c_w}s_w^2\gamma^5 P_L Z_{(n)5}l_{(n)}\\
&+\bar{l}_{(0)}i\frac{e_5}{\sqrt{2\pi R}}\gamma^5 P_L A_{(n)5}l_{(n)}+\bar{l}_{(n)}i\gamma^\mu P_R\partial_\mu l_{(n)}\\
&+\bar{l}_{(n)}\frac{g_{w5}/\sqrt{2\pi R}}{c_w}s_w^2\gamma^\mu P_R Z_{(0)\mu}l_{(n)}+\bar{l}_{(n)}\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_R A_{(n)\mu}l_{(0)}\\
&+\bar{l}_{(n)}\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_R A_{(0)\mu}l_{(n)}+\bar{l}_{(m)}e_5\gamma^\mu P_R A_{(k)\mu}\Delta_{nmk}l_{(n)}\\
&+\bar{l}_{(n)}P_L\frac{n}{R}l_{(n)}-\bar{l}_{(m)}i\frac{g_{w5}}{c_w}s_w^2 P_L Z_{(k)5}\Delta_{nk,m}l_{(n)}\\
&-\bar{l}_{(m)}ie_5 P_L A_{(k)5}\Delta_{nk,m}l_{(n)}+\bar{l}_{(n)}i\gamma^\mu P_L\partial_\mu l_{(n)}\\
&+\bar{l}_{(n)}\frac{g_{w5}/\sqrt{2\pi R}}{c_w}s_w^2\gamma^\mu P_L Z_{(0)\mu}l_{(n)}+\bar{l}_{(m)}\frac{g_{w5}}{c_w}s_w^2\gamma^\mu P_L Z_{(k)\mu}\Delta_{nm,k}l_{(n)}\\
&+\bar{l}_{(n)}\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu P_L A_{(0)\mu}l_{(n)}+\bar{l}_{(m)}e_5\gamma^\mu P_L A_{(k)\mu}\Delta_{nm,k}l_{(n)}\\
&+\bar{l}_{(n)}P_R\frac{n}{R}l_{(n)}+\bar{l}_{(n)}i\frac{g_{w5}/\sqrt{2\pi R}}{c_w}s_w^2 P_R Z_{(n)5}l_{(0)}\\
&+\bar{l}_{(m)}i\frac{g_{w5}}{c_w}s_w^2 P_R Z_{(k)5}\Delta_{mk,n}l_{(n)}+\bar{l}_{(n)}i\frac{e_5}{\sqrt{2\pi R}}P_R A_{(n)5}l_{(0)}\\
&+\bar{l}_{(m)}ie_5 P_R A_{(k)5}\Delta_{mk,n}l_{(n)}
\end{aligned}
$$

We recover the massless singlet field[15]:

$$
\begin{aligned}
\mathscr{L}_l \xrightarrow{\int_0^{2\pi R}dy} \cdots &\supset \bar{l}_{(0)}\left(i\gamma^\mu\partial_\mu+\frac{g_{w5}/\sqrt{2\pi R}}{c_w}s_w^2\gamma^\mu Z_{(0)\mu}+\frac{e_5}{\sqrt{2\pi R}}\gamma^\mu A_{(0)\mu}\right)l_{(0)}\\
&\supset \bar{l}_{(0)}i\gamma^\mu\left(\partial_\mu-i\frac{g_w}{c_w}s_w^2 Z_{(0)\mu}-ieA_{(0)\mu}\right)l_{(0)}
\end{aligned}
$$

By matching the gauge couplings with those in the 4D theory, we make the identification

$$
\begin{aligned}
e &= \frac{e_5}{\sqrt{2\pi R}}\\
g_w &= \frac{g_{w5}}{\sqrt{2\pi R}}
\end{aligned}
$$

so the gauge couplings in a theory with extra dimensions depend on the radius of compactification.

As for the extra-dimensional singlet, the mass term does not come out with the correct sign:

$$
\begin{aligned}
\mathscr{L}_l \xrightarrow{\int_0^{2\pi R}dy} \cdots &\supset \bar{l}_{(n)}\left(\frac{n}{R}P_R+\frac{n}{R}P_L\right)l_{(n)}\\
&\supset \bar{l}_{(n)}\frac{n}{R}l_{(n)}
\end{aligned}
$$

Fortunately, this is not as wild as it seems: this is a *bug* that pops up in extra-dimensional theories [23]. The problem arises because of the mismatch of some quantum numbers between the 4- and the 5D theory. However, it can be overcome, and we may safely treat it as a mass term.

[15] Again, the fermion masses derive from the Yukawa Lagrangian once the symmetry is spontaneously broken. By not obtaining a mass term at the zero order as expected, we are on the right track.

Meantime, we obtain some precious Feynman rules for the fermion singlet[16]. Those which are of immediate interest to us – the fermion-fermion-vector vertices with indices $f_{(0)}f_{(n)}V_{(n)}$ – are summarized at Section 4.

3.3.3 The lepton doublet

In this section, an analysis for extracting the vertex factors for fermion-fermion-vector interactions will be performed by using the lepton doublet. We keep the charge matrix implicit so that we will be able to study the quark doublet if desired.

The fermion Lagrangian contains the kinetic term for the lepton doublet as

$$
\begin{aligned}
\mathscr{L}_{\text{fermion}} &\supset \bar{\mathcal{L}} i \Gamma^M \mathscr{D}^{\mathcal{L}}_M \mathcal{L} \\
&\supset \bar{\mathcal{L}} i \left(\Gamma^\mu \mathscr{D}^{\mathcal{L}}_\mu + \Gamma^5 \mathscr{D}^{\mathcal{L}}_5\right) \mathcal{L} \\
&\supset \bar{\mathcal{L}} i \left(\gamma^\mu \mathscr{D}^{\mathcal{L}}_\mu + i\gamma^5 \mathscr{D}^{\mathcal{L}}_5\right) \left(\frac{1}{\sqrt{2\pi R}} L_{(0)} + P_L L_{(n)} c_n + P_R L_{(n)} s_n\right) \\
&\supset \bar{\mathcal{L}} \Bigg(\frac{1}{\sqrt{2\pi R}} i\gamma^\mu \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(0)} + i\gamma^\mu P_L \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(n)} c_n + i\gamma^\mu P_R \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(n)} s_n \\
&\qquad - \frac{1}{\sqrt{2\pi R}} \gamma^5 \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(0)} + P_L \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} c_n - P_R \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} s_n\Bigg) \\
&\supset \left(\frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} + \overline{P_L \mathcal{L}_{(m)}} c_m + \overline{P_R \mathcal{L}_{(m)}} s_m\right) \Bigg(\frac{1}{\sqrt{2\pi R}} i\gamma^\mu \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(0)} \\
&\qquad + i\gamma^\mu P_L \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(n)} c_n + i\gamma^\mu P_R \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(n)} s_n - \frac{1}{\sqrt{2\pi R}} \gamma^5 \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(0)} \\
&\qquad + P_L \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} c_n - P_R \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} s_n\Bigg) \\
&\supset \left(\frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} + \bar{\mathcal{L}}_{(m)} P_R c_m + \bar{\mathcal{L}}_{(m)} P_L s_m\right) \Bigg(\frac{1}{\sqrt{2\pi R}} i\gamma^\mu \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(0)} \\
&\qquad + i\gamma^\mu P_L \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(n)} c_n + i\gamma^\mu P_R \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(n)} s_n - \frac{1}{\sqrt{2\pi R}} \gamma^5 \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(0)} \\
&\qquad + P_L \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} c_n - P_R \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} s_n\Bigg) \\
&\supset \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} i\gamma^\mu P_L \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(n)} c_n + \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} i\gamma^\mu P_R \mathscr{D}_\mu \mathcal{L}_{(n)} s_n \\
&\qquad + \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} P_L \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} c_n - \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} P_R \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(n)} s_n \\
&\qquad + \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(m)} i\gamma^\mu P_L \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(0)} c_m - \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(m)} P_R c_m \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(0)} \\
&\qquad + \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(m)} i\gamma^\mu P_R \mathscr{D}^{\mathcal{L}}_\mu \mathcal{L}_{(0)} s_m + \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(m)} P_L s_m \mathscr{D}^{\mathcal{L}}_5 \mathcal{L}_{(0)} \\
&\supset \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} i\gamma^\mu P_L \Bigg[-\frac{i g_{w5}}{\sqrt{2}} \left(\tau^+ W^+_{(k)\mu} + \tau^- W^-_{(k)\mu}\right)
\end{aligned}
$$

[16]If we let $Z_M \to -QZ_M$ and $A_M \to -QA_M$, then the above analysis will also cover the quark singlet.

$$
\begin{aligned}
&- \frac{ig_{w5}}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(k)\mu} + ie_5 Q A_{(k)\mu}\Bigg] c_k \mathcal{L}_{(n)} c_n \\
&- \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(0)} P_R \Bigg[- \frac{ig_{w5}}{\sqrt{2}}\left(\tau^+ W^+_{(k)5} + \tau^- W^-_{(k)5}\right) \\
&- \frac{ig_{w5}}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(k)5} + ie_5 Q A_{(k)5}\Bigg] s_k \mathcal{L}_{(n)} s_n \\
&+ \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(m)} i\gamma^\mu P_L \Bigg[- \frac{ig_{w5}}{\sqrt{2}}\left(\tau^+ W^+_{(k)\mu} + \tau^- W^-_{(k)\mu}\right) \\
&- \frac{ig_{w5}}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(k)\mu} + ie_5 Q A_{(k)\mu}\Bigg] c_k \mathcal{L}_{(0)} c_m \\
&+ \frac{1}{\sqrt{2\pi R}} \bar{\mathcal{L}}_{(m)} P_L s_m \Bigg[- \frac{ig_{w5}}{\sqrt{2}}\left(\tau^+ W^+_{(k)5} + \tau^- W^-_{(k)5}\right) \\
&- \frac{ig_{w5}}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(k)5} + ie_5 Q A_{(k)5}\Bigg] s_k \mathcal{L}_{(0)} \\
&\xrightarrow{\int_0^{2\pi R} dy} \bar{\mathcal{L}}_{(0)} i\gamma^\mu P_L \Bigg[- \frac{ig_w}{\sqrt{2}}\left(\tau^+ W^+_{(n)\mu} + \tau^- W^-_{(n)\mu}\right) \\
&- \frac{ig_w}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(n)\mu} + ieQ A_{(n)\mu}\Bigg] L_{(n)} \\
&- \bar{\mathcal{L}}_{(0)} P_R \Bigg[- \frac{ig_w}{\sqrt{2}}\left(\tau^+ W^+_{(n)5} + \tau^- W^-_{(n)5}\right) - \frac{ig_w}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(n)5} \\
&+ ieQ A_{(n)5}\Bigg] \mathcal{L}_{(n)} \\
&+ \bar{\mathcal{L}}_{(n)} i\gamma^\mu P_L \Bigg[- \frac{ig_w}{\sqrt{2}}\left(\tau^+ W^+_{(n)\mu} + \tau^- W^-_{(n)\mu}\right) - \frac{ig_w}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(n)\mu} \\
&+ ieQ A_{(n)\mu}\Bigg] \mathcal{L}_{(0)} \\
&+ \bar{\mathcal{L}}_{(n)} P_L \Bigg[- \frac{ig_w}{\sqrt{2}}\left(\tau^+ W^+_{(n)5} + \tau^- W^-_{(n)5}\right) - \frac{ig_w}{c_w}\left(\frac{\tau^3}{2} - Qs_w^2\right) Z_{(n)5} \\
&+ ieQ A_{(n)5}\Bigg] \mathcal{L}_{(0)}
\end{aligned}
$$

For the lepton doublet, we have

$$\mathcal{L} = \begin{pmatrix} \nu_l \\ l \end{pmatrix}_L$$

and

$$Q = \begin{pmatrix} 0 & \\ & -1 \end{pmatrix}$$

and hence

$$
\begin{aligned}
\mathscr{L}_{\text{fermion}} \xrightarrow{\int_0^{2\pi R} dy} \cdots \supset & \begin{pmatrix} \bar{\nu}_{l(0)} & \bar{l}_{(0)} \end{pmatrix}_L i\gamma^\mu P_L \\
& \cdot \begin{pmatrix} -\frac{ig_w}{2c_w} Z_{(n)\mu} & -\frac{ig_w}{\sqrt{2}} W^+_{(n)\mu} \\ -\frac{ig_w}{\sqrt{2}} W^-_{(n)\mu} & -ieA_{(n)\mu} - \frac{ig_w}{c_w}\left(-\frac{1}{2}+s_w^2\right) Z_{(n)\mu} \end{pmatrix} \begin{pmatrix} \nu_{l(n)} \\ l_{(n)} \end{pmatrix}_L \\
& - \begin{pmatrix} \bar{\nu}_{l(0)} & \bar{l}_{(0)} \end{pmatrix}_L P_R \\
& \cdot \begin{pmatrix} -\frac{ig_w}{2c_w} Z_{(n)5} & -\frac{ig_w}{\sqrt{2}} W^+_{(n)5} \\ -\frac{ig_w}{\sqrt{2}} W^-_{(n)5} & -ieA_{(n)5} - \frac{ig_w}{c_w}\left(-\frac{1}{2}+s_w^2\right) Z_{(n)5} \end{pmatrix} \begin{pmatrix} \nu_{l(n)} \\ l_{(n)} \end{pmatrix}_L \\
& + \begin{pmatrix} \bar{\nu}_{l(n)} & \bar{l}_{(n)} \end{pmatrix}_L i\gamma^\mu P_L \\
& \cdot \begin{pmatrix} -\frac{ig_w}{2c_w} Z_{(n)\mu} & -\frac{ig_w}{\sqrt{2}} W^+_{(n)\mu} \\ -\frac{ig_w}{\sqrt{2}} W^-_{(n)\mu} & -ieA_{(n)\mu} - \frac{ig_w}{c_w}\left(-\frac{1}{2}+s_w^2\right) Z_{(n)\mu} \end{pmatrix} \begin{pmatrix} \nu_{l(0)} \\ l_{(0)} \end{pmatrix}_L \\
& + \begin{pmatrix} \bar{\nu}_{l(n)} & \bar{l}_{(n)} \end{pmatrix}_L P_L \\
& \cdot \begin{pmatrix} -\frac{ig_w}{2c_w} Z_{(n)5} & -\frac{ig_w}{\sqrt{2}} W^+_{(n)5} \\ -\frac{ig_w}{\sqrt{2}} W^-_{(n)5} & -ieA_{(n)5} - \frac{ig_w}{c_w}\left(-\frac{1}{2}+s_w^2\right) Z_{(n)5} \end{pmatrix} \begin{pmatrix} \nu_{l(0)} \\ l_{(0)} \end{pmatrix}_L
\end{aligned}
$$

After these expansions, it is easy to read off the vertex factors for the interactions $F_{(0)}F_{(n)}V_{(n)}$. Parts of the results are summarized in Section 4.

3.4 The Yukawa sector

The Yukawa sector proves to be the most cumbersome one of all the fields[17] This is primarily because the leptons and quarks mix among each other. We need to diagonalize the Yukawa interaction matrices appropriately so that the diagonalized matrices will directly give the mass terms, and as a by product we read off the fermion-fermion-Higgs interactions [26]. Still, the work done speaks more than the words, and we proceed by recalling the Yukawa Lagrangian.

$$
\mathscr{L}_{\text{yukawa}} = -y_{l5}\bar{\mathcal{L}} l H - y_{d5}\bar{\mathcal{Q}} d H - y_{u5}\bar{\mathcal{Q}} u\tilde{H} + \text{h.c.}
$$

As an example, we expand the leptonic part.

$$
\begin{aligned}
\mathscr{L}_{\text{yukawa}} &\supset -y_{l5}\bar{\mathcal{L}} l H + \text{h.c.} \\
&\supset -y_{l5}\bar{\mathcal{L}}\left(\frac{1}{\sqrt{2\pi R}} l_{(0)} + P_R l_{(n)} c_n + P_L l_{(n)} s_n\right)\left(\frac{1}{\sqrt{2\pi R}} + H_{(m)} c_m\right) + \text{h.c.} \\
&\supset -y_{l5}\left(\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(0)} + \bar{\mathcal{L}}_{(k)} P_R c_k + \bar{\mathcal{L}}_{(k)} P_L s_k\right) \\
&\quad \times \left(\frac{1}{2\pi R} l_{(0)} H_{(0)} + \frac{1}{\sqrt{2\pi R}} l_{(0)} H_{(m)} c_m + \frac{1}{\sqrt{2\pi R}} P_R l_{(n)} H_{(0)} c_n \right.\\
&\quad \left. + P_R l_{(n)} H_{(m)} c_n c_m + \frac{1}{\sqrt{2\pi R}} P_L l_{(n)} H_{(0)} s_n + P_L l_{(n)} H_{(m)} s_n c_m\right) + \text{h.c.} \\
&\supset -y_{l5}\left(\frac{1}{(2\pi R)^{3/2}}\bar{\mathcal{L}}_{(0)} l_{(0)} H_{(0)} + \frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(0)} P_R l_{(n)} H_{(m)} c_n c_m\right.
\end{aligned}
$$

[17] Well, not actually. We have not mentioned a single word about the ghosts since they are not of primary interest to us at the moment. In order to extract the Feynman rules for the ghosts, we need the transform rules of the gauge-fixing terms, which is lengthy even in the 4D theory [16].

$$
\begin{aligned}
&+\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(k)}P_R l_{(0)}H_{(m)}c_k c_m+\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(k)}l_{(n)}H_{(0)}c_k c_n\\
&+\bar{\mathcal{L}}_{(k)}P_R l_{(n)}H_{(m)}c_k c_n c_m+\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(k)}P_L l_{(n)}H_{(0)}s_k s_n\\
&+\bar{\mathcal{L}}_{(k)}P_L l_{(n)}H_{(m)}s_k s_n c_m\Bigg)+\text{h.c.}
\end{aligned}
$$

$$
\begin{aligned}
\xrightarrow{\int_0^{2\pi R}dy}-y_{l5}\Bigg(&\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(0)}l_{(0)}H_{(0)}+\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(0)}P_R l_{(n)}H_{(n)}\\
&+\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(n)}P_R l_{(0)}H_{(n)}+\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(n)}P_R l_{(n)}H_{(0)}\\
&+\bar{\mathcal{L}}_{(k)}P_R l_{(n)}H_{(m)}\Delta_{nmk}+\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{L}}_{(n)}P_L l_{(n)}H_{(0)}\\
&+\bar{\mathcal{L}}_{(k)}P_L l_{(n)}H_{(m)}\Delta_{nm,k}\Bigg)+\text{h.c.}
\end{aligned}
$$

From the first term, we see that the Yukawa constants are also scaled by the factor $1/\sqrt{2\pi R}$ when we reduce the theory to 4D:

$$
y_j=\frac{y_{j5}}{\sqrt{2\pi R}},\qquad j=l,d,u
$$

4 Results

In this section, we present the results covering the gauge couplings and masses obtained by coefficient-matching between the 4- and the 5D theory, and some Feynman rules that we could catch due to time constrictions. Our results match with our primary resources in the literature[21, 22, 23].

4.1 Gauge coupling, masses, and other constants

Seemingly, the gauge couplings in 5D are matched with their counterparts in 4D, up to a factor $1/\sqrt{2\pi R}$[18]:

$$
g_c=\frac{g_{c5}}{\sqrt{2\pi R}}
$$

$$
g_w=\frac{g_{w5}}{\sqrt{2\pi R}}
$$

$$
e=\frac{e_5}{\sqrt{2\pi R}}
$$

[18]The gauge couplings are dimensionless in 4D, which can be seen by comparing the Lagrangians for the photon and the fermion fields. First, let us state a relation of the basic units of measurement: Since $c=1$, we have $[x]=[t]$ and $[E]=[m]$, and since $\hbar=1$, we write $[E]=1/[t]$. Therefore, the integral measure in 4D has mass dimension -4, so the Lagrangians has mass dimensions 4. The kinetic term for the photon have mass dimension $[\partial^2A^2]=m^2[A]^2=m^4$ so $[A]=m$. From the fermion-photon interaction and the fermion mass term, respectively, we write $[\psi gA\psi]=[\psi m\psi]$ and therefore $[g]=1$. However, we see that the gauge couplings receive mass dimension $[x]^{1/2}=1/[m]^{1/2}$, which is the source of the non-renormalizability of the 5D theory.

The Yukawa constants also receive the same factor:

$$y_j = \frac{y_{j5}}{\sqrt{2\pi R}}$$

What we may call the *Higgs coefficients* are adjusted a bit differently,

$$\mu = \mu_5$$

$$\lambda = \frac{\lambda_5}{2\pi R}$$

so is the $1/\sqrt{2\pi R}$-factor for the Higgs VEV:

$$v = v_5\sqrt{2\pi R}$$

where $v = \sqrt{\mu^2/\lambda}$.

The fields that appear in the KK tower emerge with a common mass term, $m_n = n/R$, therefore the overall mass for a field $\mathcal{K}$ can be written as

$$M_{\mathcal{K}} = \sqrt{m_{\mathcal{K}}^2 + \frac{n^2}{R^2}}$$

where $m_{\mathcal{K}}$ is the 4D-mass of the field $\mathcal{K}$ that arise due to spontaneous symmetry breaking as usual.

4.2 Conservation of KK number

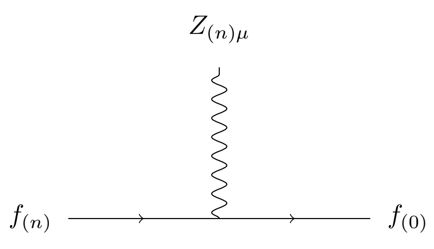
The KK number is conserved solely because all the SM fields live in the bulk. The two laws for the conservation of KK number can be summarized as follows:

1. At each vertex, the sum of KK numbers towards the vertex equals the sum of those outwards the vertex.
2. The states with odd KK numbers exist in pair; that is, for each KK field with odd KK number at a vertex, there should appear another KK field with odd KK number.

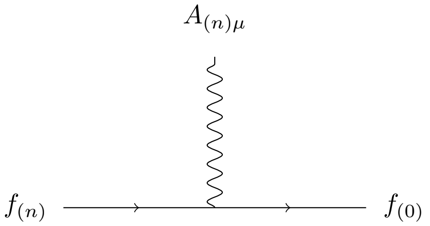
4.3 Feynman rules

4.3.1 ffV vertex

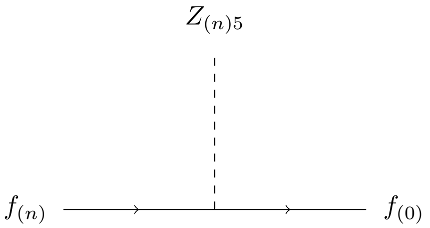
In this section, we present parts of the Feynman rules for the singlet fermions and gauge bosons.



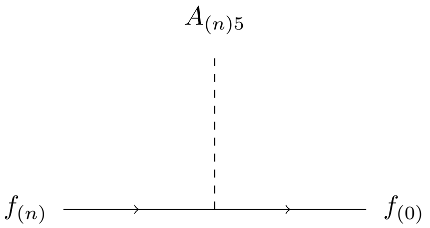
$$-\frac{ig_w}{c_w} s_w^2 \gamma^\mu P_R Q_f$$



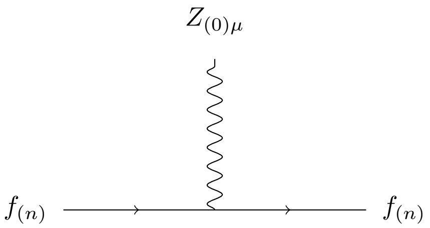
$$-ie\gamma^\mu P_R Q_f$$



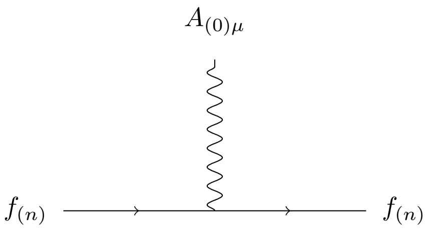
$$\frac{g_w}{c_w} s_w^2 \gamma^5 P_L Q_f$$



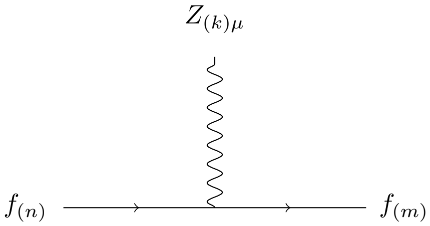
$$e\gamma^5 Q_f$$



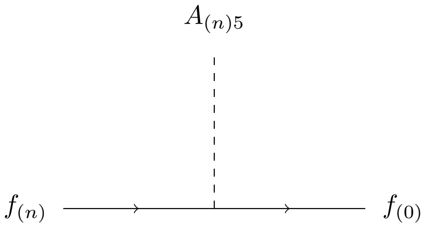
$$-\frac{ig_w}{c_w} s_w^2 \gamma^\mu Q_f$$



$$-ie\gamma^\mu Q_f$$



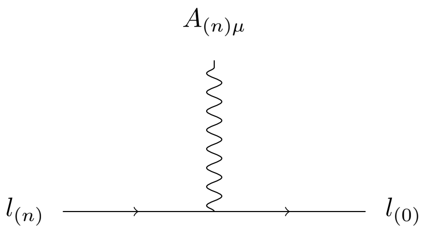
$$-\frac{ig_{w5}}{c_w}s_w^2\gamma^\mu P_L\Delta_{nm,k}$$



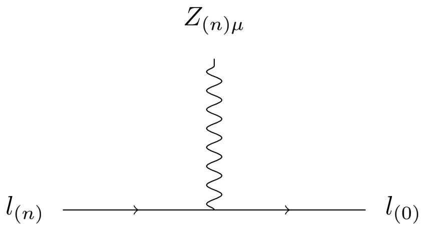
$$-ie_5\gamma^\mu P_L\Delta_{nm,k}$$

4.3.2 FFV vertex

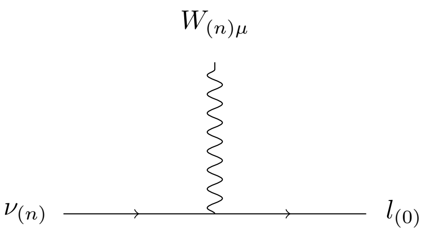
In this section, we present fragments of the Feynman rules for the doublet fermion and gauge bosons. Of our immediate interest are the vertices of the form $F_{(n)}F_0V_{(n)}$.



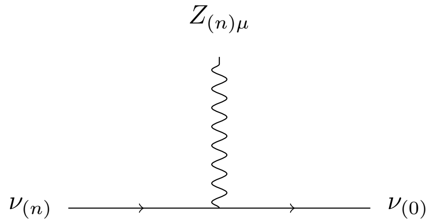
$$ie\gamma^\mu P_L$$



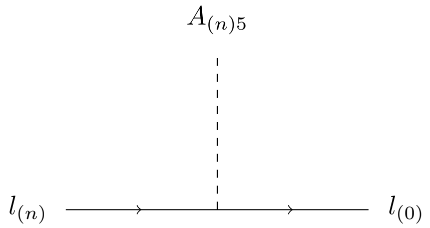
$$\frac{ig_w}{c_w}\gamma^\mu\left(-\frac{1}{2}+s_w^2\right)P_L$$



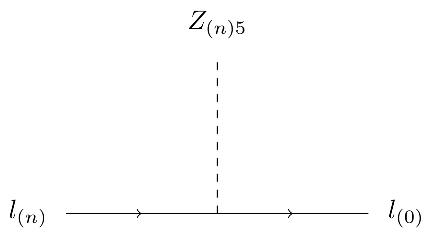
$$\frac{ig_w}{\sqrt{2}}\gamma^\mu P_L$$



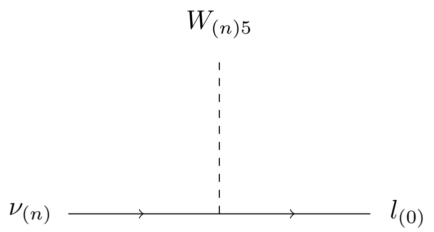
$$\frac{ig_w}{2c_w}\gamma^\mu P_L$$



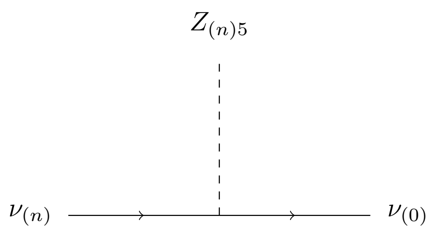
$$-eP_R$$



$$\frac{g_w}{c_w}\left(\frac{1}{2} - s_w^2\right) P_R$$



$$-\frac{g_w}{\sqrt{2}}P_R$$



$$-\frac{g_w}{2c_w}P_R$$

5 Conclusion and acknowledgement

In this project work, I tried to explain the necessity to study the Standard Model of particle physics in higher than four spacetime dimensions, and introduced the models developed outside the realm of superstring and supergravity theories. The reasons why we need to study may be summarized as follows:

1. The so-called grand unification scheme or the grand unification theory (GUT) seems to be achievable only in theories with extra dimensions. To quote a result from the string theory, an extra dimension of length $R \sim (10^{12}\,\text{GeV})^{-1}$ has been realized to bring down the string scale to the GUT scale, $M_{\text{GUT}} \sim 10^{16}\,\text{GeV}$, which will carry us to a unification of all the forces in the nature at the same scale [27, 28].

2. The hierarchy problem in the Standard Model appears to have a resolution, provided we promote the theory to higher dimensions.

3. Within the framework of the Standard Model, none has yet to show the existence of a particle such as dark matter. This may be possible in higher-dimensional scenarios.

Between 1998 and 2001, there appeared three models piquing the curiosity of phenomenological high-energy physics:

1. The ADD model (1998), named after Arkani-Hamed, Dimopoulos, and Dvali, also known as the model of large extra dimensions: It was originally proposed to provide an explanation for the large order of difference between the electroweak and Planck scales by introducing extra dimensions in which only the gravity lives. It does so because specifically in large extra dimensions, the strength of gravity is somewhat reduced [14].

2. The RS model (1999), named after Randall and Sundrum, also known as the model of warped extra dimensions: It heavily involves a modification of flat extra dimensions into warped ones, by introducing a warp factor. It was originally proposed to account for the hierarchy problem in the SM in new approaches.

3. The UED model (2001), advocated by Appelquist, Cheng, and Dobrescu, the model in which all the SM fields live in the bulk of spacetime; that is, unlike the previous two models where only gravity survives in the extra dimensions, this model allows all the matter, gauge, and Higgs fields live in all the dimensions out there. Again, unlike the two previous proposals, this model promises not to bring about a violation of the global symmetries of the SM since these symmetries constitute its backbone.

What is important here is, unlike superstring and supergravity theories, the extra-dimensional theories summarized here yields measurable outcomes, starting from the next generation colliders.

The methodology involved in promoting the Standard Model to five dimensions is quite comprehensible to follow. The usual Lorentz indices are allowed to take on an extra value, 5. Since the universe is a cylindrical one, that is to say, the extra dimension is defined on a circle, we may Fourier-expand the fields along this new spatial component. This will permit us to easily integrate out the fifth dimension and obtain an effective theory. We should note that the five-dimensional theory is non-renormalizable, which is simply because the couplings have a dependence on the radius of circle on which the fifth dimension is compactified. Since we have a dimensionful quantity that will determine the fate of the theory, there should appear a cut-off scale, Λ. Currently, physicists of the field who have studied the vacuum stability infer that $\Lambda R \sim 6$ [25], which in turn determines the maximum number of the Kaluza-Klein states to be used in the calculation of the physical quantities as $n = 6$.

Despite the *fact* that the analysis of the Standard Model in five dimensions is quite straightforward in terms of extracting the Feynman rules – at least the vertices – it clearly takes more than it is planned to. Having promoted the theory to five dimensions, I quickly realized that I needed a superior computational power to merely perform the expansion of the products of the fields. For instance, I studied the complete four-dimensional theory over a weekend, and observed that the Higgs Lagrangian alone produces around 150 terms, which I collected and evaluated by using MATHEMATICA. Accordingly, when we make a harmonic expansion of the fields, this number goes up to 1400. Clearly, this is an illusion. My *beginner* knowledge of the mentioned software was hardly sufficient to even collect the summations with similar dummy indices; hence, the computer could not realize for example that $\mathcal{K}_{1(n)}\mathcal{K}_{2(m)}c_n c_m$ and $\mathcal{K}_{1(k)}\mathcal{K}_{2(l)}c_k c_l$ refer to the same mathematical object. As an acknowledgement, I regret having had to give up to build a complete model of the five-dimensional Standard Model after struggling with becoming adept at MATHEMATICA and settle for what seems to be completed within the time domain.

Care must be provided when we expand the field in Fourier series. The very first objective is to recover the actual fields at the zero mode. This drives us to assign an odd parity to the fifth components of the vector fields under Z_2. Intriguingly, we also obtain what we *leftleft*-, *leftright*-, *rightleft*-, and *rightright*-chiral fermions within the Kaluza-Klein states. This seems to be the only working way to obtain chiral fermions in a theory with odd number of spacetime dimensions. Another aspect of the theory that deserves additional attention is the fermion masses: it comes out in the Kaluza-Klein tower with the wrong sign. At this point, we are lead to calculate the loop corrections to the masses and obtain a correct expression for the mass terms.

Towards the end of the semester, my objective has become extracting the Feynman rules for the interactions of the top quark with the charge and neutral scalars in the theory. After all, this was supposed to be one of my two motivations to study the extra dimensions, the other being the investigation of the Higgs physics in extra dimensions. There are rare top decays, which is almost non-permitted in the four-dimensional Standard Model. The theories with extra dimensions provides new aspects to relax these suppressions. I could not make it on time for that.

Working on this project has inspired me to conduct a more profound research on the subject of perhaps a *generalized* extra-dimensional theory – such as the ADD or RS models – for my graduate thesis. I focused on the *universal* extra dimensions purely because the physics was quite comprehensible.

I sincerely thank my advisor Assoc. Prof. İsmail Turan for his introducing the field of extra-dimensional theories in the context of particle physics to me and giving me a direction in my future career as a researcher in theoretical particle physics. As always, I believe I genuinely attempted to overperform the expectations of me, yet this might not have yielded the desired results as anticipated within the framework of this course.

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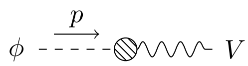
A Finding the Goldstone bosons

Finding the Goldstone bosons may be quite challenging. In the 4D theory, it is rather straightforward: we expand the Higgs Lagrangian and collect the unphysical terms – terms that are the product of a gauge boson and the 4-gradient of a scalar introduced in the Higgs doublet. However, in theories beyond Standard Model, things may get complicated, which is the case for the 5D extension of the Standard Model. Thus, we seek for a general method to aid us to find the Goldstone bosons, simply the *Goldstones*. Logan offers three most general ways to obtain these Goldstones [26], which at least works for the usual 4D theory and the 2-Higgs doublet model (2HDM). Logan favors one method more, in which she advises us to

1. write down the kinetic term for the Higgs doublet.
2. isolate the terms of the form $gvV\partial\phi$ (Lorentz indices suppressed) where g is the gauge coupling, v is the Higgs VEV (or VEVs if we study for example 2HDM), V is the gauge boson that couples with g, and ϕ is a scalar that appears in the Higgs doublet.
3. observe the unphysical vertices such as

   and infer that ϕ is – must be – the Goldstone: it is massless, and it interacts with the gauge boson in a quite unusual way.

We would like to extend this argument: in completing the picture depicted by Logan, we

4. find a term that will cancel $gvV\partial\phi$: clearly, $gv\phi\partial V$ will do by forming a total derivative,

   $$gvV\partial\phi + gv\phi\partial V = \partial\left(gv\phi V\right)$$

   to the existence of which the physics is inert at the action level.
5. find a gauge-fixing Lagrangian that may produce the this term,

   $$\begin{aligned}\mathscr{L}_{\text{gf}} &\supset gv\phi\partial V \\ &\supset -\frac{1}{2}\left(\partial V - gv\phi\right)^2\end{aligned}$$

   meantime associating gv most probably with the 4D – real-world – mass, m_V, of the gauge boson.
6. read the Goldstone as this additional term, ϕ, up to a sign.
   (a) If the term $gvV\partial\phi$ has a plus sign in the Higgs Lagrangian, then take $G = \phi$.
   (b) If it has a minus sign, then take $G = -\phi$.

Logan's method works like a charm in the 4D theory and in the 5D theory, as well.

Since the literature is abundant in the derivations of the Goldstones in 4D, we proceed to directly study the 5D case. We start with the Higgs doublet:

$$H = \begin{pmatrix} \phi_+ \\ \frac{\tilde{h}+i\phi_Z}{\sqrt{2}} \end{pmatrix}$$

Different from our earlier notation, here $\tilde{h}$ is the 5D Higgs field that contains the 4D Higgs VEV. As stated earlier, if $R^{-1} \gg v$, then only the zeroth mode of h will receive a non-vanishing VEV:

$$\begin{aligned}
\tilde{h} &= \frac{1}{\sqrt{2\pi R}}\tilde{h}_{(0)} + \tilde{h}_{(n)} c_n \\
&= \frac{1}{\sqrt{2\pi R}}\left(h_{(0)} + v\right) + h_{(n)} c_n \\
&= h + \frac{v}{\sqrt{2\pi R}}
\end{aligned}$$

But $v = v_5\sqrt{2\pi R}$ as shown earlier, so we may write

$$H = \begin{pmatrix} \phi_+ \\ \frac{h+v_5+i\phi_Z}{\sqrt{2}} \end{pmatrix}$$

(Unlike the earlier analysis, we explicitly need a VEV to begin with.) The kinetic term for the Higgs doublet is

$$\mathscr{L}_{\text{higgs}} \supset \left|\mathscr{D}_M H\right|^2$$

where, as earlier,

$$\begin{aligned}
\mathscr{D}_M &= \partial_M \mathbb{1}_2 - \frac{i g_{w5}}{\sqrt{2}} \begin{pmatrix} & W_M^+ \\ W_M^- & \end{pmatrix} - \frac{i g_{w5}}{c_w} \begin{pmatrix} \frac{1}{2} - s_w^2 & \\ & -\frac{1}{2} \end{pmatrix} Z_M + i e_5 \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} A_M \\
&= \begin{pmatrix} \partial_M - \frac{i g_{w5}}{c_w}\left(\frac{1}{2} - s_w^2\right) Z_M + i e_5 A_M & -\frac{i g_{w5}}{\sqrt{2}} W_M^+ \\ -\frac{i g_{w5}}{\sqrt{2}} W_M^- & \partial_M + \frac{i g_{w5}}{c_w}\frac{1}{2} Z_M \end{pmatrix}
\end{aligned}$$

so

$$\mathscr{D}_M H = \begin{pmatrix} \partial_M \phi_+ - \frac{i g_{w5}}{c_w}\left(\frac{1}{2} - s_w^2\right) Z_M \phi_+ + i e_5 A_M \phi_+ - \frac{i g_{w5}}{\sqrt{2}} W_M^+ \frac{h+v_5+i\phi_Z}{\sqrt{2}} \\ -\frac{i g_{w5}}{\sqrt{2}} W_M^- \phi_+ + \frac{\partial_M h + i \partial_M \phi_Z}{\sqrt{2}} + \frac{i g_{w5}}{c_w}\frac{1}{2} Z_M \frac{h+v_5+i\phi_Z}{\sqrt{2}} \end{pmatrix}$$

with

$$\left(\mathscr{D}_M H\right)^* = \begin{pmatrix} \partial_M \phi_- + \frac{i g_{w5}}{c_w}\left(\frac{1}{2} - s_w^2\right) Z_M \phi_- - i e_5 A_M \phi_- + \frac{i g_{w5}}{\sqrt{2}} W_M^- \frac{h+v_5-i\phi_Z}{\sqrt{2}} \\ \frac{i g_{w5}}{\sqrt{2}} W_M^+ \phi_- + \frac{\partial_M h - i \partial_M \phi_Z}{\sqrt{2}} - \frac{i g_{w5}}{c_w}\frac{1}{2} Z_M \frac{h+v_5-i\phi_Z}{\sqrt{2}} \end{pmatrix}$$

First of all, let us determine the 4D gauge-boson masses: The mass of the $W^\pm$ bosons is the square root of the coefficient of the term $W_\mu^+ W_\mu^-$, for which we have

$$\frac{g_{w5} v_5}{4} W_M^+ W_M^- \supset \frac{g_{w5} v_5}{4} W_\mu^+ W_\mu^- = \frac{g_w v}{4} W_\mu^+ W_\mu^-$$

since $g_{w5} = g_w\sqrt{2\pi R}$. Thus we have

$$m_W = \frac{g_w v}{2}$$

Similarly, by considering the mass terms $\frac{1}{2} m_Z^2 \left(Z_\mu\right)^2$ for the Z boson, we see that

$$m_Z = \frac{g_w v}{2 c_w}$$

We would like to preserve the letter M for the 5D effective masses. Now we continue our analysis: let us read off the terms of the form $gvV\partial\phi$:

$$\begin{aligned}\mathscr{L}_{\text{higgs}} &\supset \frac{ig_{w5}v_5}{2}W^{-M}\partial_M\phi_+ - \frac{ig_{w5}v_5}{2}W^{+M}\partial_M\phi_- + m_Z Z^M\partial_M\phi_Z \\ &\supset \left[-im_W W^{+M}\partial_M\phi_- + \text{h.c.}\right] + m_Z Z^M\partial_M\phi_Z\end{aligned}$$

The suitable gauge-fixing terms are

$$\mathscr{L}_{\text{gf}} \supset \left[-im_W\phi_-\partial_M W^{+M} + \text{h.c.}\right] + m_Z\phi_Z\partial_M Z^M$$

which mas possibly derive from

$$\mathscr{L}_{\text{gf}} \supset -\left|\partial_M W^{+M} - im_W\phi_+\right|^2 - \frac{1}{2}\left(\partial_M Z^M - m_Z\phi_Z\right)^2$$

But this is the usual gauge-fixing Lagrangian in the Feynman-'t Hooft gauge. Now let us read the Goldstones. We have actually a few steps to take: first of all, we need to extract the fifth components of the vectors since they constitute new scalars in the theory:

$$\begin{aligned}\mathscr{L}_{\text{gf}} &\supset -\left|\partial_\mu W^{+\mu} + \partial_5 W^{+5} - im_W\phi_+\right|^2 - \frac{1}{2}\left(\partial_\mu Z^\mu + \partial_5 Z^5 - m_Z\phi_Z\right)^2 \\ &\supset -\left|\partial_\mu W^{+\mu} - \partial_5 W_5^+ - im_W\phi^+\right|^2 - \frac{1}{2}\left(\partial_\mu Z^\mu - \partial_5 Z_5 - m_Z\phi_Z\right)^2\end{aligned}$$

The Goldstone for the Z boson

According to our prescription based on Logan's, $\partial_5 Z_5 + m_Z\phi_Z$ is a fine candidate:

$$\begin{aligned}M_Z G_Z &= \partial_5 Z_5 + m_Z\phi_Z \\ &\supset \partial_5\left(Z_{(n)5}s_n\right) + m_Z\phi_{Z(n)}c_n \\ &\supset \left(\frac{n}{R}Z_{(n)5} + m_Z\phi_{Z(n)}\right)c_n\end{aligned}$$

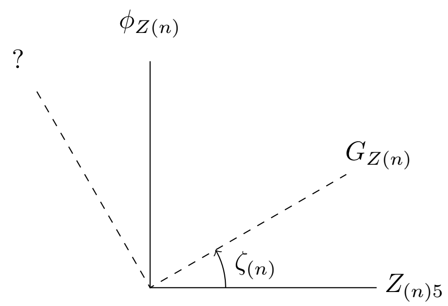
Thus we have

$$G_{Z(n)} = \frac{\frac{n}{R}Z_{(n)5} + m_Z\phi_{Z(n)}}{M_Z}$$

Now, what might M_Z be? If we perceive $Z_{(n)5}$ and $\phi_{Z(n)}$ as normalized basis vectors, then M_Z is the apparent normalization factor

$$M_Z = \sqrt{\frac{n^2}{R^2} + m_Z^2}$$

Let us elaborate on this basis issue:



So, where is the state orthogonal to $G_{Z(n)}$? Let us define

$$\cos\zeta_{(n)} := \frac{n/R}{M_Z}, \qquad \sin\zeta_{(n)} := \frac{m_Z}{M_Z}$$

Accordingly,

$$G_{Z(n)} = Z_{(n)5}\cos\zeta_{(n)} + \phi_{Z(n)}\sin\zeta_{(n)}$$

Then the orthogonal state may directly be written as

$$\begin{aligned}\chi_{Z(n)} &= -Z_{(n)5}\sin\zeta_{(n)} + \phi_{Z(n)}\cos\zeta_{(n)} \\ &= \frac{-m_Z Z_{(n)5} + \frac{n}{R}\phi_{Z(n)}}{M_Z}\end{aligned}$$

The $\chi_{Z(n)}$ are the physical, CP-odd scalars.

The Goldstone for the W^+ boson

According to our prescription based on Logan's, $-\partial_5 W_5^+ - im_W\phi_+$ is an appropriate candidate.

$$\begin{aligned}M_W G_+ &= -\partial_5 W_5^+ - im_W\phi_+ \\ &\supset -\partial_5\left(W^+_{(n)5}s_n\right) - im_W\phi_{+(n)}c_n \\ &\supset \left(-\frac{n}{R}W^+_{(n)5} - im_W\phi_{+(n)}\right)c_n\end{aligned}$$

Hence we obtain

$$G_{+(n)} = \frac{-\frac{n}{R}W^+_{(n)5} - im_W\phi_{+(n)}}{M_W}$$

Similarly to the Z-case, we have

$$M_W = \sqrt{\frac{n^2}{R^2} + m_W^2}$$

Since the basis transformation is complex, this time the rotation will be unitary:

$$\begin{pmatrix} G_{+(n)} \\ H_{+(n)} \end{pmatrix} = \mathcal{R}\begin{pmatrix} W^+_{(n)5} \\ \phi_{+(n)} \end{pmatrix}$$

where

$$\mathcal{R} := \frac{1}{M_W}\begin{pmatrix} -\frac{n}{R} & -im_W \\ \alpha & \beta \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{C}$. To satisfy unitarity, we have

$$\begin{aligned}&\mathcal{R}^\dagger\mathcal{R} = \mathbb{1}_2 \\ &\frac{1}{M_W^2}\begin{pmatrix} -\frac{n}{R} & \alpha^* \\ im_W & \beta^* \end{pmatrix}\begin{pmatrix} -\frac{n}{R} & -im_W \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

$$\frac{1}{M_W^2}\begin{pmatrix} \frac{n^2}{R^2}+|\alpha|^2 & \frac{im_W n}{R}+\alpha^*\beta \\ -\frac{im_W n}{R}+\beta^*\alpha & m_W^2+|\beta|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, one set of solutions contains

$$\alpha = \pm i m_W, \qquad \beta = \pm\frac{n}{R}$$

We may choose the plus signs and learn to live with it[19]:

$$H_{+(n)} = \frac{i m_W W^+_{(n)5} + \frac{n}{R}\phi_{+(n)}}{M_W}$$

The Goldstone for the W^- boson

This is just the complex conjugate of G^+:

$$G_{-(n)} = \frac{-\frac{n}{R}W^-_{(n)5} + i m_W \phi_{-(n)}}{M_W}$$

and accordingly

$$H_{-(n)} = \frac{-i m_W W^-_{(n)5} + \frac{n}{R}\phi_{-(n)}}{M_W}$$

Summary

The Goldstone bosons that live in the KK tower of the 5D theory have been found to be

$$G_{Z(n)} = \frac{\frac{n}{R}Z_{(n)5} + m_Z\phi_{Z(n)}}{\sqrt{\frac{n^2}{R^2}+m_Z^2}} = \frac{m_n Z_{(n)5} + m_Z\phi_{Z(n)}}{M_Z}$$

$$G_{\pm(n)} = \frac{-\frac{n}{R}W^\pm_{(n)5} \mp i m_W\phi_{\pm(n)}}{\sqrt{\frac{n^2}{R^2}+m_W^2}} = \frac{-m_n W^\pm_{(n)5} \mp i m_W\phi_{\pm(n)}}{M_W}$$

The physical, CP-odd scalars are

$$\chi_{Z(n)} = \frac{-m_Z Z_{(n)5} + \frac{n}{R}\phi_{Z(n)}}{\sqrt{\frac{n^2}{R^2}+m_Z^2}} = \frac{-m_Z Z_{(n)5} + m_n\phi_{Z(n)}}{M_Z}$$

The physical, charged Higgs bosons are

$$H_{\pm(n)} = \frac{\pm i m_W W^\pm_{(n)5} + \frac{n}{R}\phi_{\pm(n)}}{\sqrt{\frac{n^2}{R^2}+m_W^2}} = \frac{\pm i m_W W^\pm_{(n)5} + m_n\phi_{\pm(n)}}{M_W}$$

[19] If we had taken the minus signs, the only difference we foresee would be in the sign of the vertex factors.

B Fermion-scalar interaction

B.1 Sample calculations for the top quark interactions

In this section, we revisit the Yukawa sector to extract the Feynman rules for the vertices of the form $F_{(0)}F_{(n)}S_{(n)}$ where the $F_{(0)}$ is a real-world fermion – without any explicit chirality – and $S_{(n)}$ is a KK scalar. Our primary objective was to investigate the effects of the KK states on top physics, therefore we focus our utmost attention on the vertices that contain interactions of the real-world top quark, t. However, due to possible ambiguities in the notation, we would like to denote the SM top quark by $\mathbf{t}_{(0)}$ since $\mathcal{T}_{(0)}$ denotes the left-chiral 4D top quark and $t_{(0)}$ the right-chiral one.

We proceed by recalling the Yukawa Lagrangian:

$$\mathscr{L}_{\text{yukawa}} \supset -y_{u5}\bar{\mathcal{Q}}u\tilde{H} - y_{d5}\bar{\mathcal{Q}}dH + \text{h.c.} \qquad \text{(sum over generations)}$$

Let us directly start with the diagonalized Yukawa matrices, which implies that the down-type quarks will be Cabibbo-rotated:

$$\mathcal{Q}_i \to \mathcal{Q}'_i = V_{ij}\mathcal{Q}_j, \qquad d_i \to d'_i = V_{ij}d_j, \qquad i,j = 1,2,3$$

where V is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Clearly, it is only the bottom element of the quark doublet that is affected by this rotation:

$$\mathcal{Q}'_i = \begin{pmatrix} \mathcal{U}_i \\ \mathcal{D}'_i \end{pmatrix}$$

In this *mass* basis (as opposed to the *generation* basis), the Yukawa matrices read

$$y_{u5} = \begin{pmatrix} Y_{u5} & & \\ & Y_{c5} & \\ & & Y_{t5} \end{pmatrix}, \qquad y_{d5} = \begin{pmatrix} Y_{d5} & & \\ & Y_{s5} & \\ & & Y_{b5} \end{pmatrix}$$

Hence we have

$$\begin{aligned}
\mathscr{L}_{\text{yukawa}} \supset & -\begin{pmatrix} \bar{\mathcal{Q}}'_u & \bar{\mathcal{Q}}'_c & \bar{\mathcal{Q}}'_t \end{pmatrix} \begin{pmatrix} Y_{u5} & & \\ & Y_{c5} & \\ & & Y_{t5} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix} \tilde{H} \\
& - \begin{pmatrix} \bar{\mathcal{Q}}'_d & \bar{\mathcal{Q}}'_s & \bar{\mathcal{Q}}'_b \end{pmatrix} \begin{pmatrix} Y_{d5} & & \\ & Y_{s5} & \\ & & Y_{b5} \end{pmatrix} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} H + \text{h.c.} \\
\supset & -Y_{t5}\bar{\mathcal{Q}}'_t t\tilde{H} - Y_{b5}\bar{\mathcal{Q}}'_b b' H + \text{h.c.} \\
\supset & -\begin{pmatrix} Y_{t5}\bar{\mathcal{T}}t & Y_{t5}\bar{\mathcal{B}}'t \end{pmatrix} \begin{pmatrix} \frac{h+v_5-i\phi_Z}{\sqrt{2}} \\ \phi_- \end{pmatrix} - \begin{pmatrix} Y_{b5}\bar{\mathcal{T}}b' & Y_{b5}\bar{\mathcal{B}}'b' \end{pmatrix} \begin{pmatrix} \phi_+ \\ \frac{h+v_5+i\phi_Z}{\sqrt{2}} \end{pmatrix} + \text{h.c.} \\
\supset & -\frac{Y_{t5}v_5}{\sqrt{2}}\bar{\mathcal{T}}t - Y_{t5}\bar{\mathcal{B}}'t\phi_- - \frac{Y_{b5}v_5}{\sqrt{2}}\bar{\mathcal{B}}'b' - Y_{b5}\bar{\mathcal{T}}b'\phi_+ + \text{h.c.} \\
\supset & \left[-\frac{Y_{t5}v_5}{\sqrt{2}}\bar{\mathcal{T}}t - \frac{Y_{t5}v_5}{\sqrt{2}}\bar{t}\mathcal{T}\right] + \left[-\frac{Y_{b5}v_5}{\sqrt{2}}\bar{\mathcal{B}}'b' - \frac{Y_{b5}v_5}{\sqrt{2}}\bar{b}'\mathcal{B}'\right] + \left[-Y_{t5}\bar{\mathcal{B}}'t\phi_- - Y_{t5}\bar{t}\mathcal{B}'\phi_+\right] \\
& + \left[-Y_{b5}\bar{\mathcal{T}}b'\phi_+ - Y_{b5}\bar{b}'\mathcal{T}\phi_-\right]
\end{aligned}$$

where we have used $(\bar{a}b)^\dagger = \left(a^\dagger\gamma^0 b\right)^\dagger = b^\dagger\gamma^0 a = \bar{b}a$. Now, the first two brackets are the mass terms for the top and bottom quarks, respectively. Since we have shown that

$$y_j = \frac{y_{j5}}{\sqrt{2\pi R}}, \qquad v = v_5\sqrt{2\pi R}$$

we may directly write $Y_{t5}v_5 = Y_t v$ and $Y_{b5}v_5 = Y_b v$. Meantime, the capital-letter quarks are LH, and the lowercase-letter quarks are RH; they combine to give

$$\bar{q}_L q_R + \bar{q}_R q_L = \overline{P_L q} P_R q + \overline{P_R q} P_L q = \bar{q} P_R q + \bar{q} P_L q = \bar{q}q$$

if this were the 4D theory. Fortunately, once we expand the fields in Fourier series, we recover the 4D theory in the zeroth mode hence our discussion makes sense. Therefore we determine the quark masses to be

$$m_q = \frac{Y_q v}{\sqrt{2}}$$

where q denotes a SM quark. Now let us focus on the interaction terms:

$$\begin{aligned}
\mathscr{L}_{\text{yukawa}} &\supset \left[-Y_{t5}\bar{\mathcal{B}}' t\phi_- - Y_{t5}\bar{t}\mathcal{B}'\phi_+\right] + \left[-Y_{b5}\bar{\mathcal{T}}b'\phi_+ - Y_{b5}\bar{b}'\mathcal{T}\phi_-\right] \\
&\supset \left[-Y_{t5}V_{3j}^*\bar{\mathcal{B}}_j t\phi_- - Y_{t5}V_{3j}\bar{t}\mathcal{B}_j\phi_+\right] + \left[-Y_{b5}V_{3j}\bar{\mathcal{T}}b_j\phi_+ - Y_{b5}V_{3j}^*\bar{b}_j\mathcal{T}\phi_-\right]
\end{aligned}$$

Now we expand the fields, by keeping only the zeroth term of the top field, and the KK states of the bottom field:

$$\begin{aligned}
\mathscr{L}_{\text{yukawa}} \supset{}& -Y_{t5}V_{3j}^*\left(\overline{P_L\mathcal{B}_{(n)j}}c_n + \overline{P_R\mathcal{B}_{(n)j}}s_n\right)\left(\frac{1}{\sqrt{2\pi R}}t_{(0)}\right)\left(\phi_{-(m)}c_m\right) \\
&- Y_{t5}V_{3j}\left(\frac{1}{\sqrt{2\pi R}}\bar{t}_{(0)}\right)\left(P_L\mathcal{B}_{(n)j}c_n + P_R\mathcal{B}_{(n)j}s_n\right)\left(\phi_{+(m)}c_m\right) \\
&- Y_{b5}V_{3j}\left(\frac{1}{\sqrt{2\pi R}}\bar{\mathcal{T}}_{(0)}\right)\left[P_R b_{(n)j}c_n + P_L b_{(n)j}\right]\left(\phi_{+(m)}c_m\right) \\
&- Y_{b5}V_{3j}^*\left(\overline{P_R b_{(n)j}}c_n + \overline{P_R b_{(n)j}}s_n\right)\left(\frac{1}{\sqrt{2\pi R}}\mathcal{T}_{(0)}\right)\left(\phi_{-(m)}c_m\right) \\
\supset{}& -Y_t V_{3j}^*\bar{\mathcal{B}}_{(n)j}P_R t_{(0)}\phi_{-(m)}c_n c_m - Y_t V_{3j}\bar{t}_{(0)}P_L\mathcal{B}_{(n)j}\phi_{+(m)}c_n c_m \\
&- Y_b V_{3j}\bar{\mathcal{T}}_{(0)}P_R b_{(n)j}\phi_{+(m)}c_n c_m - Y_b V_{3j}^*\bar{b}_{(n)j}P_L\mathcal{T}_{(0)}\phi_{-(m)}c_n c_m \\
\xrightarrow{\int_0^{2\pi R}dy}{}& -Y_t V_{3j}^*\bar{\mathcal{B}}_{(n)j}P_R t_{(0)}\phi_{-(n)} - Y_t V_{3j}\bar{t}_{(0)}P_L\mathcal{B}_{(n)j}\phi_{+(n)} \\
&- Y_b V_{3j}\bar{\mathcal{T}}_{(0)}P_R b_{(n)j}\phi_{+(n)} - Y_b V_{3j}^*\bar{b}_{(n)j}P_L\mathcal{T}_{(0)}\phi_{-(n)}
\end{aligned}$$

There remains one final rotation to perform[20]: even though the Yukawa matrices seem to be diagonalized, when we expand the fields and perform the necessary integrations, we see that the down-type quarks mix with each other. The *real* KK down-type quarks then become

$$\mathcal{B}_{(n)j,L/R} = \mp\mathcal{B}'_{(n)j,L/R}\cos\alpha_{(n)} + b'_{(n)j,L/R}\sin\alpha_{(n)}$$

$$b_{(n)j,L/R} = \pm\mathcal{B}'_{(n)j,L/R}\sin\alpha_{(n)} + b'_{(n)j,L/R}\cos\alpha_{(n)}$$

where

$$\alpha_{(n)} := \frac{1}{2}\tan^{-1}\frac{m_j}{m_n}$$

[20] I promise to derive this in no time, but not quite sure now.

The chiral indices refer to the KK states $P_{L/R}\mathcal{B}_{(n)j}$ and $P_{L/R}b_{(n)j}$. It is crucial to notice that we have eliminated $\mathcal{B}_{(n)j,R}$ and $b_{(n)j,L}$ out of our discussion just because their coefficient is s_n, which cancels out c_m, the coefficient of the scalar, upon integration over y. Thus, by performing this rotation, the interaction part of the Yukawa Lagrangian becomes

$$\begin{aligned}\mathscr{L}_{\text{yukawa}} \xrightarrow{\int_0^{2\pi R} dy} \cdots \supset & -Y_t V_{3j}^* \left(-\bar{\mathcal{B}}'_{(n)j}\cos\alpha_{(n)} + b'_{(n)j}\sin\alpha_{(n)}\right) P_R t_{(0)}\phi_{-(n)} \\ & - Y_t V_{3j}\bar{t}_{(0)} P_L \left(-\bar{\mathcal{B}}'_{(n)j}\cos\alpha_{(n)} + b'_{(n)j}\sin\alpha_{(n)}\right)\phi_{+(n)} \\ & - Y_b V_{3j}\bar{\mathcal{T}}_{(0)} P_R \left(-\mathcal{B}'_{(n)j}\sin\alpha_{(n)} + b'_{(n)j}\cos\alpha_{(n)}\right)\phi_{+(n)} \\ & - Y_b V_{3j}^* \left(-\bar{\mathcal{B}}'_{(n)j}\sin\alpha_{(n)} + \bar{b}'_{(n)j}\cos\alpha_{(n)}\right) P_L \mathcal{T}_{(0)}\phi_{-(n)}\end{aligned}$$

Next, let us explicitly convert the chiral top quarks into the real one:

$$t_{(0)} = P_R\mathbf{t}_{(0)}, \qquad \mathcal{T}_{(0)} = P_L\mathbf{t}_{(0)}$$

Since $P_R^2 = P_R$, $P_L^2 = P_L$ and $\overline{P_{L/R}a} = \bar{a}P_{R/L}$, we get

$$\begin{aligned}\mathscr{L}_{\text{yukawa}} \xrightarrow{\int_0^{2\pi R} dy} \cdots \supset & -Y_t V_{3j}^* \left(-\bar{\mathcal{B}}'_{(n)j}\cos\alpha_{(n)} + b'_{(n)j}\sin\alpha_{(n)}\right) P_R \mathbf{t}_{(0)}\phi_{-(n)} \\ & - Y_b V_{3j}^* \left(-\bar{\mathcal{B}}'_{(n)j}\sin\alpha_{(n)} + \bar{b}'_{(n)j}\cos\alpha_{(n)}\right) P_L \mathbf{t}_{(0)}\phi_{-(n)} \\ & - Y_t V_{3j}\bar{\mathbf{t}}_{(0)} P_L \left(-\bar{\mathcal{B}}'_{(n)j}\cos\alpha_{(n)} + b'_{(n)j}\sin\alpha_{(n)}\right)\phi_{+(n)} \\ & - Y_b V_{3j}\bar{\mathbf{t}}_{(0)} P_R \left(-\mathcal{B}'_{(n)j}\sin\alpha_{(n)} + b'_{(n)j}\cos\alpha_{(n)}\right)\phi_{+(n)}\end{aligned}$$

Ultimately, we write the scalars $\phi_{\pm(n)}$ in terms of the Goldstones and charged Higgs bosons: The unitary rotation

$$\begin{pmatrix} G_{\pm(n)} \\ H_{\pm(n)} \end{pmatrix} = \frac{1}{M_W}\begin{pmatrix} -m_n & \mp i m_W \\ \pm i m_W & m_n \end{pmatrix}\begin{pmatrix} W^{\pm}_{(n)5} \\ \phi_{\pm(n)} \end{pmatrix}$$

can be inverted to yield

$$\begin{pmatrix} W^{\pm}_{(n)5} \\ \phi_{\pm(n)} \end{pmatrix} = \frac{1}{M_W}\begin{pmatrix} -m_n & \mp i m_W \\ \pm i m_W & m_n \end{pmatrix}\begin{pmatrix} G_{\pm(n)} \\ H_{\pm(n)} \end{pmatrix}$$

and hence

$$\phi_{\pm(n)} = \frac{\pm i m_W G_{\pm(n)} + m_n H_{\pm(n)}}{M_W}$$

Thus, by writing the Yukawa couplings in terms of the 4D masses

$$Y_{t/d} = \frac{m_{t/d}\sqrt{2}}{v}$$

and the gauge boson masses in terms of the gauge coupling and the Higgs VEV

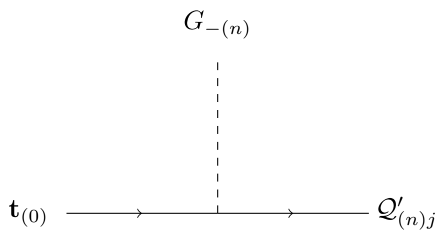
$$m_W = \frac{g_w v}{2}$$

we obtain

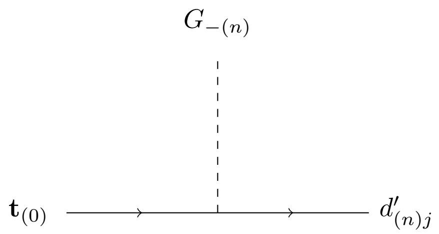
$$
\begin{aligned}
\mathscr{L}_{\text{yukawa}} \xrightarrow{\int_0^{2\pi R} dy} \cdots \supset & -\frac{m_t\sqrt{2}}{v} V_{3j}^* \left(-\bar{\mathcal{B}}'_{(n)j} \cos\alpha_{(n)} + \bar{b}'_{(n)j} \sin\alpha_{(n)}\right) P_R \mathbf{t}_{(0)} \frac{-i\frac{g_w v}{2} G_{-(n)} + m_n H_{-(n)}}{M_W} \\
& - \frac{m_b\sqrt{2}}{v} V_{3j}^* \left(-\bar{\mathcal{B}}'_{(n)j} \sin\alpha_{(n)} + \bar{b}'_{(n)j} \cos\alpha_{(n)}\right) P_L \mathbf{t}_{(0)} \frac{-i\frac{g_w v}{2} G_{-(n)} + m_n H_{-(n)}}{M_W} \\
& - \frac{m_t\sqrt{2}}{v} V_{3j} \bar{\mathbf{t}}_{(0)} P_L \left(-\mathcal{B}'_{(n)j} \cos\alpha_{(n)} + b'_{(n)j} \sin\alpha_{(n)}\right) \frac{i\frac{g_w v}{2} G_{+(n)} + m_n H_{+(n)}}{M_W} \\
& - \frac{m_b\sqrt{2}}{v} V_{3j} \bar{\mathbf{t}}_{(0)} P_R \left(-\mathcal{B}'_{(n)j} \sin\alpha_{(n)} + b'_{(n)j} \cos\alpha_{(n)}\right) \frac{i\frac{g_w v}{2} G_{+(n)} + m_n H_{+(n)}}{M_W}
\end{aligned}
$$

B.2 Feynman rules

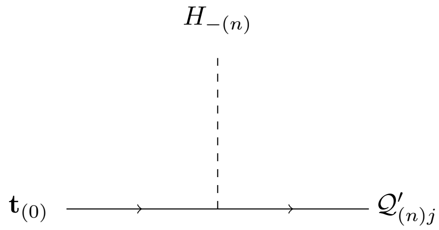
First of all, let us realize that the quarks $b'_{(n)j}$ are simply $d'_{(n)}$, $s'_{(n)}$, and $b'_{(n)}$, therefore we may use the generic notation, d, for this term. Similarly, let us switch to the notation $\mathcal{Q}$ for what is represented by $\mathcal{B}$. This is all thanks to a summation over the above-mentioned quarks with the CKM matrix. As for the vertices that involve $H_{\pm(n)}$, to eliminate the Higgs VEV v we multiply and divide by $m_W = g_w v/2$. At the end of the day, we produce the following diagrams:



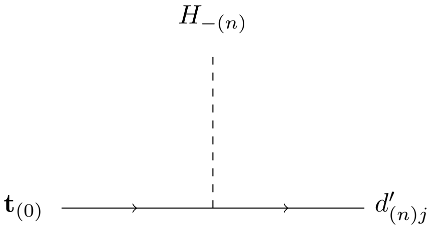
$$\frac{g_w}{\sqrt{2}M_W}\left(m_t \cos\alpha_{(n)} P_R + m_j \sin\alpha_{(n)} P_L\right) V_{3j}^*$$



$$-\frac{g_w}{\sqrt{2}M_W}\left(m_t \sin\alpha_{(n)} P_R + m_j \cos\alpha_{(n)} P_L\right) V_{3j}^*$$



$$\frac{ig_w}{\sqrt{2}M_W}\frac{m_n}{m_w}\left(m_t \cos\alpha_{(n)} P_R + m_j \sin\alpha_{(n)} P_L\right) V_{3j}^*$$



$$\frac{ig_w}{\sqrt{2}M_W}\frac{m_n}{m_w}\left(m_t \sin\alpha_{(n)} P_R + m_j \cos\alpha_{(n)} P_L\right) V_{3j}^*$$

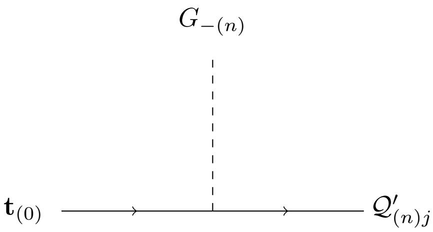
If we note that

1. the complex conjugate of these expressions give the vertices for $\bar{\mathbf{t}}_{(0)}\{G_{\pm(n)}, H_{\pm(n)}\}\{\mathcal{Q}'_{(n)j}, d'_{(n)j}\}$,
2. if we release the subscript 3 into a generalized index, $\mathbf{i}$, then our diagrams hold true also for the SM quarks $\mathbf{u}_{(0)}$ and $\mathbf{d}_{(0)}$,

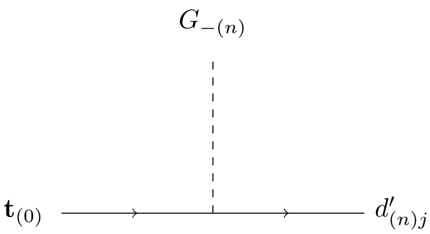
hence we do not repeat the diagrams.

B.3 Comparison with the results in the literature

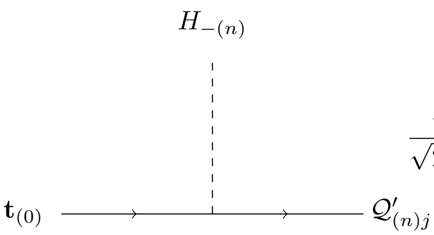
Our primary source for the fermion-scalar vertices is Reference [25]. Their results, adjusted to our notation, are as follows:



$$\frac{g_w}{\sqrt{2}M_W}\left[-m_t \cos\alpha_{(n)} P_R + \left(m_j \sin\alpha_{(n)} + m_n \cos\alpha_{(n)}\right) P_L\right] V_{3j}^*$$



$$-\frac{g_w}{\sqrt{2}M_W}\left[m_t \sin\alpha_{(n)} P_R - \left(m_j \cos\alpha_{(n)} - m_n \sin\alpha_{(n)}\right) P_L\right] V_{3j}^*$$



$$\frac{ig_w}{\sqrt{2}M_W}\left[-\frac{m_t m_n}{m_w}\cos\alpha_{(n)} P_R + \left(\frac{m_j m_n}{m_w}\sin\alpha_{(n)} - m_w \cos\alpha_{(n)}\right) P_L\right] V_{3j}^*$$

C Fermion mixing in the KK tower

Let us work with the leptons in the mass basis. When we have to do a similar analysis for the quarks in the mass basis, we simply tack in a factor V_{ij}, where V is the CKM matrix.

C.1 Fermion sector revisited

We proceed by writing down the fermion Lagrangian and extracting only the mass terms, since the particles will mix in terms of their masses.

$$\begin{aligned}
\mathscr{L}_{\text{fermion}} &\supset \bar{\mathcal{L}} i\Gamma^M \mathscr{D}_M \mathcal{L} + \bar{l} i \Gamma^M \mathscr{D}_M l, \qquad \Gamma^M = \left(\gamma^\mu, i\gamma^5\right) \\
&\supset -\bar{\mathcal{L}}\gamma^5 \partial_5 \mathcal{L} - \bar{l}\gamma^5 \partial_5 l \\
&\supset -\left(\bar{\mathcal{L}}_{(n)L} c_n + \bar{\mathcal{L}}_{(n)R} s_n\right)\gamma^5 \partial_5 \left(\mathcal{L}_{(m)L} c_m + \mathcal{L}_{(m)R} s_m\right) \\
&\qquad - \left(\bar{l}_{(n)R} c_n + \bar{l}_{(n)L} s_n\right)\gamma^5 \partial_5 \left(l_{(m)R} c_m + l_{(m)L} s_m\right), \qquad \begin{Bmatrix} \mathcal{L} \\ l \end{Bmatrix}_{(n)R/L} := P_{R/L} \begin{Bmatrix} \mathcal{L} \\ l \end{Bmatrix}_{(n)} \\
&\supset -\left(\bar{\mathcal{L}}_{(n)L} c_n + \bar{\mathcal{L}}_{(n)R} s_n\right)\gamma^5 \left(-\frac{m}{R}\mathcal{L}_{(m)L} s_m + \frac{m}{R}\mathcal{L}_{(m)R} c_m\right) \\
&\qquad - \left(\bar{l}_{(n)R} c_n + \bar{l}_{(n)L} s_n\right)\gamma^5 \left(-\frac{m}{R} l_{(m)R} s_m + \frac{m}{R} l_{(m)L} c_m\right) \\
&\supset -\left(\bar{\mathcal{L}}_{(n)L} c_n \gamma^5 \frac{m}{R}\mathcal{L}_{(m)R} c_m - \bar{\mathcal{L}}_{(n)R} s_n \gamma^5 \frac{m}{R}\mathcal{L}_{(m)L} s_m\right) \\
&\qquad - \left(\bar{l}_{(n)R} c_n \gamma^5 \frac{m}{R} l_{(m)L} c_m - \bar{l}_{(n)L} s_n \gamma^5 \frac{m}{R} l_{(m)R} s_m\right) \\
&\xrightarrow{\int_0^{2\pi R} dy} \cdots \supset -\left(m_n \bar{\mathcal{L}}_{(n)L}\gamma^5 \mathcal{L}_{(n)R} - m_n \bar{\mathcal{L}}_{(n)R}\gamma^5 \mathcal{L}_{(n)L}\right) - \left(m_n \bar{l}_{(n)R}\gamma^5 l_{(n)L} - m_n \bar{l}_{(n)L}\gamma^5 l_{(n)R}\right) \\
&\supset \begin{pmatrix} \bar{\mathcal{L}}_{(n)L} & \bar{l}_{(n)L} \end{pmatrix} \begin{pmatrix} -m_n & 0 \\ 0 & m_n \end{pmatrix} \gamma^5 \begin{pmatrix} \mathcal{L}_{(n)R} \\ l_{(n)R} \end{pmatrix} + \begin{pmatrix} \bar{\mathcal{L}}_{(n)R} & \bar{l}_{(n)R} \end{pmatrix} \begin{pmatrix} m_n & 0 \\ 0 & -m_n \end{pmatrix} \gamma^5 \begin{pmatrix} \mathcal{L}_{(n)L} \\ l_{(n)L} \end{pmatrix}
\end{aligned}$$

We have to eliminate the gamma matrix in between. The cleanest way to do so is to refer to the projection operators:

$$\left.\begin{aligned} P_R &= \frac{1+\gamma^5}{2} \\ P_L &= \frac{1-\gamma^5}{2} \end{aligned}\right\} \gamma^5 = P_R - P_L$$

Fortunately, one of the two operators in the gamma matrix drops and the other gives 1; to illustrate,

$$\gamma^5 \begin{pmatrix} \mathcal{L}_{(n)R} \\ l_{(n)R} \end{pmatrix} = (P_R - P_L) \begin{pmatrix} \mathcal{L}_{(n)R} \\ l_{(n)R} \end{pmatrix} = (1-0) \begin{pmatrix} \mathcal{L}_{(n)R} \\ l_{(n)R} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{(n)R} \\ l_{(n)R} \end{pmatrix}$$

Thus we have

$$\mathscr{L}_{\text{fermion}} \xrightarrow{\int_0^{2\pi R} dy} \cdots \supset -\begin{pmatrix} \bar{\mathcal{L}}_{(n)L} & \bar{l}_{(n)L} \end{pmatrix} \begin{pmatrix} m_n & 0 \\ 0 & -m_n \end{pmatrix} \begin{pmatrix} \mathcal{L}_{(n)R} \\ l_{(n)R} \end{pmatrix} + \text{h.c.}$$

since the mass matrix will receive a minus sign when the doublet $\begin{pmatrix} \mathcal{L}_{(n)L} & l_{(n)L} \end{pmatrix}^T$ is acted on by the gamma matrix. After this point, we may directly read off the mixing of the bottom element of the lepton doublet, to with the left-chiral electron-type leptons, with the lepton singlet. This is necessary since it is in this format that the Yukawa sector produces the mixing terms.

C.2 Yukawa sector revisited

Again, we proceed by writing down the Yukawa Lagrangian and extract only the terms that contain mass.

$$
\begin{aligned}
\mathscr{L}_{\text{yukawa}} &\supset -y_{l5}\bar{\mathcal{L}}lH + \text{h.c.} \\
&\supset -y_{l5}\left(\bar{L}_{(n)L}c_n + \bar{\mathcal{L}}_{(n)R}s_n\right)\left(l_{(m)R}c_m + l_{(m)L}s_m\right)\left(\frac{1}{\sqrt{2\pi R}}H_{(0)}\right) + \text{h.c.} \\
&\supset -y_l\left(\bar{\mathcal{L}}_{(n)L}c_n l_{(m)R}c_m + \bar{\mathcal{L}}_{(n)R}s_n l_{(m)L}s_m\right)H_{(0)} + \text{h.c.} \\
&\xrightarrow{\int_0^{2\pi R} dy} \cdots \supset -y_l\left(\bar{\mathcal{L}}_{(n)L}l_{(n)R} + \bar{\mathcal{L}}_{(n)R}l_{(n)L}\right)H_{(0)} + \text{h.c.} \\
&\supset -\left(\bar{\mathcal{L}}_{(n)L}y_l l_{(n)R} + \bar{\mathcal{L}}_{(n)R}y_l l_{(n)L}\right)H_{(0)} + \text{h.c.} \\
&\supset -\Bigg[\begin{pmatrix}\bar{\mathcal{L}}^1_{(n)L} & \bar{\mathcal{L}}^2_{(n)L} & \bar{\mathcal{L}}^3_{(n)L}\end{pmatrix}\begin{pmatrix}Y_l^1 & & \\ & Y_l^2 & \\ & & Y_l^3\end{pmatrix}\begin{pmatrix}l^1_{(n)R} \\ l^2_{(n)R} \\ l^3_{(n)R}\end{pmatrix} \\
&\qquad + \begin{pmatrix}\bar{\mathcal{L}}^1_{(n)R} & \bar{\mathcal{L}}^2_{(n)R} & \bar{\mathcal{L}}^3_{(n)R}\end{pmatrix}\begin{pmatrix}Y_l^1 & & \\ & Y_l^2 & \\ & & Y_l^3\end{pmatrix}\begin{pmatrix}l^1_{(n)L} & l^2_{(n)L} & l^3_{(n)L}\end{pmatrix}\Bigg]H_{(0)} + \text{h.c.} \\
&\supset -\left(Y_l^\alpha\bar{\mathcal{L}}^\alpha_{(n)L}l^\alpha_{(n)R} + Y_l^\alpha\bar{\mathcal{L}}^\alpha_{(n)R}l^\alpha_{(n)L}\right)H_{(0)} + \text{h.c.} \\
&\supset -Y_l^\alpha\begin{pmatrix}\bar{\mathcal{N}}^\alpha_{(n)L} & \bar{\mathcal{E}}^\alpha_{(n)L}\end{pmatrix}l^\alpha_{(n)R}\begin{pmatrix}\phi_{+(0)} \\ \dfrac{h_{(0)} + v + i\phi_{Z(0)}}{\sqrt{2}}\end{pmatrix} \\
&\qquad - Y_l^\alpha\begin{pmatrix}\bar{\mathcal{N}}^\alpha_{(n)R} & \bar{\mathcal{E}}^\alpha_{(n)R}\end{pmatrix}l^\alpha_{(n)L}\begin{pmatrix}\phi_{+(0)} \\ \dfrac{h_{(0)} + v + i\phi_{Z(0)}}{\sqrt{2}}\end{pmatrix} + \text{h.c.} \\
&\supset -\frac{Y_l^\alpha v}{\sqrt{2}}\bar{\mathcal{E}}^\alpha_{(n)L}l^\alpha_{(n)R} - \frac{Y_l^\alpha v}{\sqrt{2}}\bar{\mathcal{E}}^\alpha_{(n)R}l^\alpha_{(n)} - \frac{Y_l^\alpha v}{\sqrt{2}}\bar{l}^\alpha_{(n)R}\mathcal{E}^\alpha_{(n)L} - \frac{Y_l^\alpha v}{\sqrt{2}}\bar{l}^\alpha_{(n)L}\mathcal{E}^\alpha_{(n)R} \\
&\supset -m_l^\alpha\bar{\mathcal{E}}^\alpha_{(n)L}l^\alpha_{(n)R} - m_l^\alpha\bar{\mathcal{E}}^\alpha_{(n)R}l^\alpha_{(n)} - m_l^\alpha\bar{l}^\alpha_{(n)R}\mathcal{E}^\alpha_{(n)L} - m_l^\alpha\bar{l}^\alpha_{(n)L}\mathcal{E}^\alpha_{(n)R} \\
&\supset \begin{pmatrix}\bar{\mathcal{E}}^\alpha_{(n)L} & \bar{l}^\alpha_{(n)L}\end{pmatrix}\begin{pmatrix}0 & -m_l^i \\ -m_l^i & 0\end{pmatrix}\begin{pmatrix}\mathcal{E}^\alpha_{(n)R} \\ l^\alpha_{(n)R}\end{pmatrix} + \begin{pmatrix}\bar{\mathcal{E}}^\alpha_{(n)R} & \bar{l}^\alpha_{(n)R}\end{pmatrix}\begin{pmatrix}0 & -m_l^i \\ -m_l^i & 0\end{pmatrix}\begin{pmatrix}\mathcal{E}^\alpha_{(n)L} \\ l^\alpha_{(n)L}\end{pmatrix}
\end{aligned}
$$

where

$\alpha = 1, 2, 3 =$ generation index

$\mathcal{N}, \mathcal{E} =$ neutrino- and electron-type lepton in the doublet, respectively

and there is a summation implied over the generation index.

C.3 The combined term

Totally, after we extract the bottom elements of the doublets, we obtain

$$
\mathscr{L}_{\text{fermion}} + \mathscr{L}_{\text{yukawa}} \xrightarrow{\int_0^{2\pi R} dy} \cdots \supset -\begin{pmatrix}\bar{\mathcal{E}}^\alpha_{(n)L} & \bar{l}^\alpha_{(n)L}\end{pmatrix}\begin{pmatrix}m_n & m_l^\alpha \\ m_l^\alpha & -m_n\end{pmatrix}\begin{pmatrix}\mathcal{E}^\alpha_{(n)R} \\ l^\alpha_{(n)R}\end{pmatrix} + \text{h.c.} \tag{C.1}
$$

and this is the fermion mixing in the KK tower. The mass matrix

$$\Lambda := \begin{pmatrix} m_n & m_l^\alpha \\ m_l^\alpha & -m_n \end{pmatrix}$$

can be diagonalized by using the similarity transformation

$$\Lambda_* = D\Lambda D^\dagger$$

where

$$\Lambda_* = \begin{pmatrix} M_l^\alpha & \\ & -M_l^\alpha \end{pmatrix}$$

$$D = \begin{pmatrix} \cos\frac{\delta_{(n)}}{2} & \sin\frac{\delta_{(n)}}{2} \\ \sin\frac{\delta_{(n)}}{2} & -\cos\frac{\delta_{(n)}}{2} \end{pmatrix}$$

with

$$M_l^\alpha := \sqrt{m_n^2 + (m_l^\alpha)^2}$$

$$\cos\delta_{(n)} = \frac{m_n}{M_l^\alpha}, \qquad \sin\delta_{(n)} = \frac{m_l^\alpha}{M_l^\alpha}$$

The mass eigenstates are then given by

$$\begin{pmatrix} \tilde{\mathcal{E}}^\alpha_{(n)L/R} \\ \tilde{l}^\alpha_{(n)L/R} \end{pmatrix} = D \begin{pmatrix} \mathcal{E}^\alpha_{(n)L/R} \\ l^\alpha_{(n)L/R} \end{pmatrix} = \begin{pmatrix} \cos\frac{\delta_{(n)}}{2} & \sin\frac{\delta_{(n)}}{2} \\ \sin\frac{\delta_{(n)}}{2} & -\cos\frac{\delta_{(n)}}{2} \end{pmatrix} \begin{pmatrix} \mathcal{E}^\alpha_{(n)L/R} \\ l^\alpha_{(n)L/R} \end{pmatrix}$$

We can see this if we rewrite (C.1) in perhaps a lousy notation as

$$\langle L \mid \Lambda \mid R\rangle = \langle L \mid D^\dagger \Lambda_* D \mid R\rangle = \langle DL \mid \Lambda_* \mid DR\rangle$$

where L (R) refers to the doublet with left- (right-) chiral leptons.

C.4 Bi-unitary transform

Even though the mass matrix Λ in the previous section is one of the simplest to obtain, and the unitary matrix D that diagonalizes it is indeed idempotent ($D^\dagger = D^{-1} = D$ so $D^2 = 1$), this should *not* be the case simply because we have different types of fermions on each side. The correct way to diagonalize Λ is then by a bi-unitary transform,

$$\Lambda_* = D_L \Lambda D_R^\dagger$$

so that D_L diagonalizes $\Lambda\Lambda^\dagger$ and D_R diagonalizes $\Lambda^\dagger\Lambda$. Then we have new matrices to unitarily transform now:

$$\Lambda_L^2 := \Lambda\Lambda^\dagger, \qquad \Lambda_R^2 := \Lambda^\dagger\Lambda$$

The former is[21]

$$\Lambda_L^2 = \begin{pmatrix} M_l^2 & \\ & M_l^2 \end{pmatrix}$$

[21] Let us suppress the generation index until the next notice.

Since the mass matrix is symmetric, the latter definition is identical to the former:

$$\Lambda_R^2 = \begin{pmatrix} M_l^2 & \\ & M_l^2 \end{pmatrix}$$

Here, the freshly defined Λ-matrices are just equal to the diagonalized mass matrix, Λ_*, squared. This implies, at least, that

$$\det D_{R/L} = \pm 1$$

Meantime, there is a counting of degree of freedom (DOF)[22]. First of all, let us start with a general 2×2 complex matrix, Ω. The number of DOF is 8. By the condition of unit determinant, this number goes down to 5. If we collect the phases in Ω, then we will be able to assign unique phases to the particles on each side, hence the DOF is counted as 1, which is a real parameter. Most probably the simplest form is then, inspired by the original form of the D matrix,

$$D_{R/L} = \begin{pmatrix} \pm\cos\frac{\delta_{(n)}}{2} & \pm\sin\frac{\delta_{(n)}}{2} \\ \pm\sin\frac{\delta_{(n)}}{2} & \pm\cos\frac{\delta_{(n)}}{2} \end{pmatrix}$$

for which the signs should be arranged to give the required determinants. One way to do is as follows:

$$D_L = \begin{pmatrix} \cos\frac{\delta_{(n)}}{2} & \sin\frac{\delta_{(n)}}{2} \\ \sin\frac{\delta_{(n)}}{2} & -\cos\frac{\delta_{(n)}}{2} \end{pmatrix}, \qquad D_R = \begin{pmatrix} \cos\frac{\delta_{(n)}}{2} & \sin\frac{\delta_{(n)}}{2} \\ -\sin\frac{\delta_{(n)}}{2} & \cos\frac{\delta_{(n)}}{2} \end{pmatrix}$$

with

$$\det D_R = +1, \qquad \det D_L = -1$$

as done in [21]. However, in [25] adjusted to our notation, the signs are a bit different:

$$D_R = \begin{pmatrix} \cos\frac{\delta_{(n)}}{2} & \sin\frac{\delta_{(n)}}{2} \\ -\sin\frac{\delta_{(n)}}{2} & \cos\frac{\delta_{(n)}}{2} \end{pmatrix}, \qquad D_L = \begin{pmatrix} -\cos\frac{\delta_{(n)}}{2} & \sin\frac{\delta_{(n)}}{2} \\ \sin\frac{\delta_{(n)}}{2} & \cos\frac{\delta_{(n)}}{2} \end{pmatrix}$$

This is not a problem since they have changed the order of the eigenvalues in the diagonalized matrix Λ_*.

[22] I have solved this problem for another course, I recall. Thus I borrow the idea but not sure for the rest.