Take $\pi = 3$, $g = 10$ m/s^2, $\sin(37°) = 0.6$, and $\cos(37°) = 0.8$. As a common knowledge, you should know the radius of the planet that you live on.

Warm up

Consider a pendulum constructed using a massless, rigid rod of length ℓ and a bob of mass m attached at its end and let the rod hang from the ceiling under the action of gravity at its other end. Suppose you give the rod an initial angle of $\theta_0 \ll 1$ rad measured from the vertical and release it from rest. Sketch the velocity and acceleration vectors of the bob when it makes $\theta = \theta_0$, $\theta = \theta_0/2$, $\theta = 0$, $\theta = -\theta_0/2$, and $\theta = -\theta_0$.

Solution • The configuration is depicted in Fig. 1.

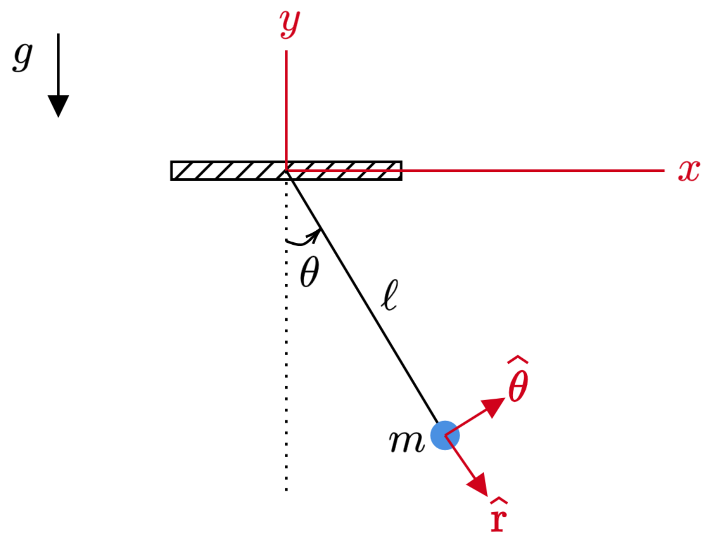

Figure 1:

The bob is set free at $\theta = \theta_0 \ll 1$ rad from rest. It will make simple harmonic motion:

$$\ddot{\theta} + \omega_0^2 \theta = 0 \tag{1}$$

where $\omega_0^2 := g/\ell$. The most general solution is

$$\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{2}$$

Let's also compute the angular velocity:

$$\omega(t) = \dot{\theta}(t) = -\omega_0 A \sin(\omega_0 t) + \omega_0 B \cos(\omega_0 t) \tag{3}$$

We are given that $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$:

$$\theta_0 = A \tag{4}$$

$$0 = \omega_0 B \Rightarrow B = 0 \tag{5}$$

Hence,

$$\theta(t) = \theta_0 \cos(\omega_0 t) \tag{6}$$

for small θ_0 and $t \geq 0$. We know the position of the bob:

$$\mathbf{r} = \ell \sin(\theta)\hat{\mathbf{x}} - \ell\cos(\theta)\hat{\mathbf{y}} \doteq \begin{pmatrix} \ell\sin(\theta) \\ -\ell\cos(\theta) \end{pmatrix} \tag{7}$$

Let's compute the velocity and acceleration:

$$\mathbf{v} = \dot{\mathbf{r}} \doteq \ell\dot{\theta}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \ell\dot{\theta}[\cos(\theta)\hat{\mathbf{x}} + \sin(\theta)\hat{\mathbf{y}}] \tag{8}$$

Notice that the unit vector in the square brackets is just $\hat{\boldsymbol{\theta}}$, which is expected even intuitively. Then,

$$\mathbf{v} = \ell\dot{\theta}\hat{\boldsymbol{\theta}} \tag{9}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ell\ddot{\theta}\hat{\boldsymbol{\theta}} + \ell\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} \tag{10}$$

$\hat{\boldsymbol{\theta}}$ is not a constant unit vector in space, unlike $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$.

$$\dot{\hat{\boldsymbol{\theta}}} \doteq \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \dot{\theta}\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} = -\dot{\theta}\sin(\theta)\hat{\mathbf{x}} + \dot{\theta}\cos(\theta)\hat{\mathbf{y}} \tag{11}$$

Note that this is $\hat{\mathbf{r}}$, which can be also expected. Then,

$$\mathbf{a} = \ell\ddot{\theta}\hat{\boldsymbol{\theta}} + \ell\dot{\theta}^2\hat{\mathbf{r}} \tag{12}$$

The acceleration has two components. The first term is the tangential one and the second the radial, as the unit vectors imply.

Let's insert the solution now. But we have to make a series expansion with respect to small θ first:

$$\mathbf{v} = \ell\dot{\theta}\hat{\boldsymbol{\theta}} \doteq \ell\dot{\theta}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \simeq \ell\dot{\theta}\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \ell\dot{\theta}\hat{\mathbf{x}} + \ell\dot{\theta}\theta\hat{\mathbf{y}} \tag{13}$$

$$\begin{aligned}\mathbf{a} &= \ell\ddot{\theta}\hat{\boldsymbol{\theta}} + \ell\dot{\theta}^2\hat{\mathbf{r}} \doteq \ell\ddot{\theta}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} + \ell\dot{\theta}^2\begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix} \simeq \ell\ddot{\theta}\begin{pmatrix} 1 \\ \theta \end{pmatrix} + \ell\dot{\theta}^2\begin{pmatrix} \theta \\ -1 \end{pmatrix} \\ &= (\ell\ddot{\theta} + \ell\dot{\theta}^2\theta)\hat{\mathbf{x}} + (\ell\ddot{\theta}\theta - \ell\dot{\theta}^2)\hat{\mathbf{y}}\end{aligned} \tag{14}$$

θ goes like θ_0 in amplitude, which is small. Moreover, $\dot{\theta}$ and $\ddot{\theta}$ also go like θ_0, multiplied by some power of ω_0. Thus, we can and should keep only the lowest power of θ_0 in an expression:

$$\mathbf{v} \simeq \ell\dot{\theta}\hat{\mathbf{x}} = -\ell\theta_0\omega_0\sin(\omega_0 t)\hat{\mathbf{x}} \tag{15}$$

$$\mathbf{a} \simeq \ell\ddot{\theta}\mathbf{x} = -\ell\theta_0\omega_0^2\cos(\omega_0 t)\hat{\mathbf{x}} \tag{16}$$

From the original solution, we have

$$\theta = \theta_0\cos(\omega_0 t) \Rightarrow t = \frac{1}{\omega_0}\arccos\left(\frac{\theta}{\theta_0}\right) \tag{17}$$

and hence

$$\mathbf{v} = -\ell\theta_0\omega_0\sin\left[\arccos\left(\frac{\theta}{\theta_0}\right)\right]\hat{\mathbf{x}} \tag{18}$$

$$\mathbf{a} = -\ell\theta_0\omega_0^2\frac{\theta}{\theta_0}\hat{\mathbf{x}} = -\ell\theta\omega_0^2\hat{\mathbf{x}} \tag{19}$$

Let's draw them now in Fig. 2.

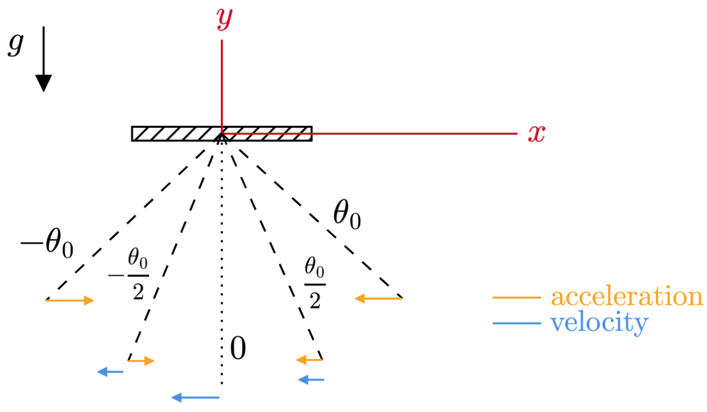

Figure 2:

But, of course, you shouldn't do it like this, taking two pages. You can make a direct analogy with the spring system because the bob makes simple harmonic motion (see Fig. 3).

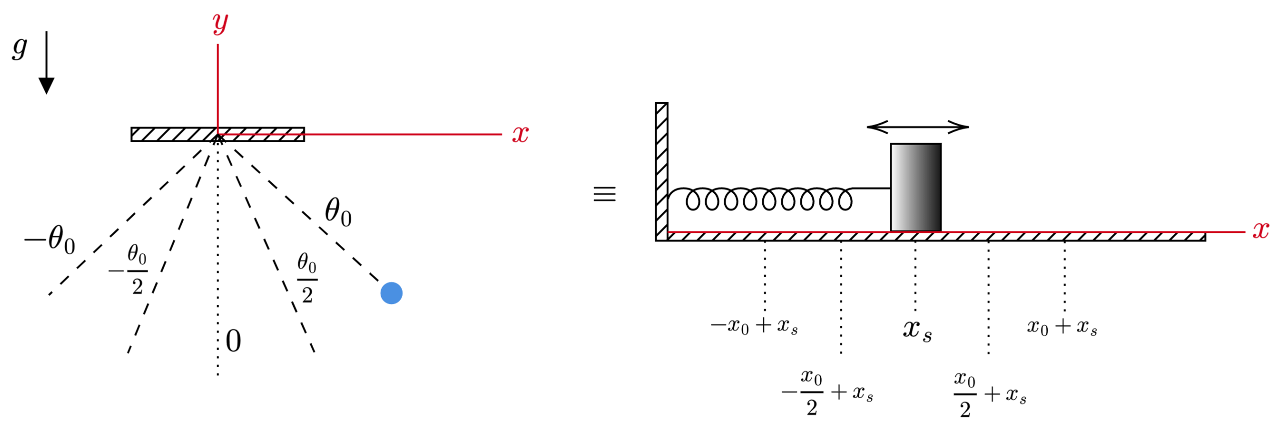

Figure 3:

Then we don't have to deal with all those vectors and rather focus on what's going on in $1D$. I don't expect you to do any rigorous math, either. Just use your physical intuition (see Fig. 4).

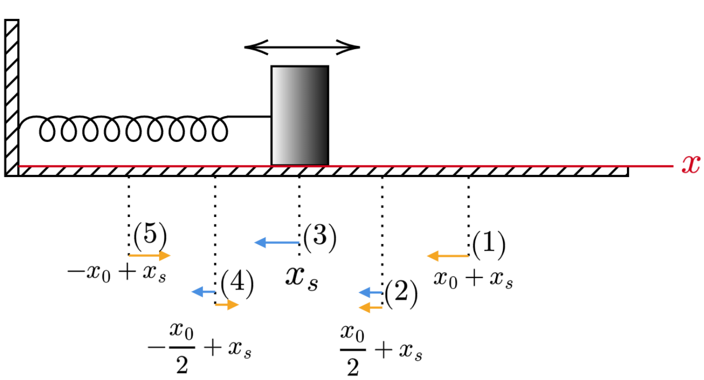

Figure 4:

Here is a summary of what's going on in Fig. 4:

(1) We give the spring an initial stretch. Hence, the restoring force will be at some maximum. This causes the maximum acceleration of the body in size throughout the motion. The spring will want to restore its initial size, so the acceleration is to the left. Also, since we release it from rest, the velocity is zero.

(2) Halfway through the initial spring length, x_s, the restoring force gets smaller because the length at which the spring is stretched is smaller, and so does the acceleration. However, all the acceleration accumulating so far increases the speed.

(3) The spring restores its initial, unstretched length. There is no force and hence no acceleration. The body cannot accelerate further so its velocity at this point should be maximum.

(4) Now we are compressing the spring, so the restoring force will want to stretch it again, causing an acceleration in the opposite direction, which will slow down the body.

(5) The body comes to a stop. The restoring force becomes maximum again, giving a full acceleration in the opposite direction.

Cardio (Problem 14-98 from Tipler's)

During an earthquake, a floor oscillates horizontally in approximately simple harmonic motion. Assume it oscillates at a single frequency with a period of 0.60 s. (a) After the earthquake, you are in charge of examining the video of the floor motion and discover that a box on the floor started to slip ben the amplitude reached 10 cm. From your data, determine the coefficient of static friction between the box and the floor. (b) If the coefficient of friction between the box and floor were 0.50, what would be the maximum amplitude of vibration before the box would slip?

Solution • The configuration is depicted in Fig. 5.

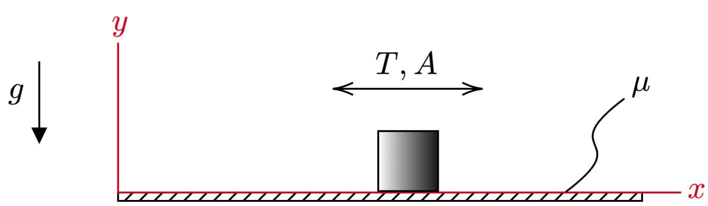

Figure 5:

The ground makes simple harmonic motion, so the box will feel it, trying to have

$$x = A\cos(\omega_0 t + \delta) \tag{20}$$

We already know this. Now, unless the acceleration is high enough, the floor won't let the box move due to friction. Let's get the acceleration:

$$v = \dot{x} = -\omega_0 A \sin(\omega_0 t + \delta) \tag{21}$$
$$a = \dot{v} = -\omega_0^2 A \cos(\omega_0 t + \delta) \tag{22}$$

Consider the maximum acceleration because it is this maximum value that needs to just overcome friction. Let's draw the free-body diagram of the box (see Fig. 6).

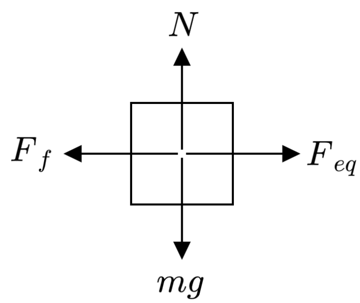

Figure 6:

Consider the motion along the y axis: There is no motion in this axis.

$$N - mg = 0 \Rightarrow N = mg \tag{23}$$

Consider the motion along the x axis:

$$F_{eq} - F_f = ma \tag{24}$$

Consider the moment $a = 0^-$, i.e. the moment just before the box start slipping. We have static friction in action:

$$F_{eq} = F_{f,\text{static}} \tag{25}$$
$$m\omega_0^2 A = \mu_s N = \mu_s mg \tag{26}$$

and hence

$$\mu_s = \frac{\omega_0^2 A}{g} = \left(\frac{2\pi}{T}\right)^2 \frac{A}{g} = \left(\frac{2\times 3}{0.60\text{ s}}\right)^2 \frac{10\times 10^{-2}\text{ m}}{10\text{ m/s}^2} \tag{27}$$
$$\boxed{\mu_s = 1.0} \tag{28}$$

In part (b), we fix the static friction coefficient and ask for the amplitude:

$$A = \frac{\mu_s g}{\omega_0^2} = \left(\frac{T}{2\pi}\right)^2 \mu_s g = \left(\frac{0.60\text{ s}}{2\times 3}\right)^2 \times 0.5 \times 10\text{ m/s}^2 \tag{29}$$
$$\boxed{A = 5.0\text{ cm}} \tag{30}$$

Core (Problem 3-67 from Tipler's)

Earth rotates on its axis once every 24 hours, so that objects on its surface execute uniform circular motion about the axis with a period of 24 hours. Consider only the effect of this rotation on the person on the surface. (Ignore Earth's orbital motion about the Sun.) (a) What is the speed and what is the magnitude of the acceleration of a person standing on the equator? (Express the magnitude of this acceleration as a percentage of g.) (b) What is the direction of the acceleration vector? (c) What is the speed and what is the magnitude of the acceleration of a person standing on the surface at 37°N latitude? (d) What is the angle between the direction of the acceleration of the person at 37°N and the direction of the acceleration of the person at the equator if both persons are at the same longitude?

Solution • The configuration is depicted in Fig. 7.

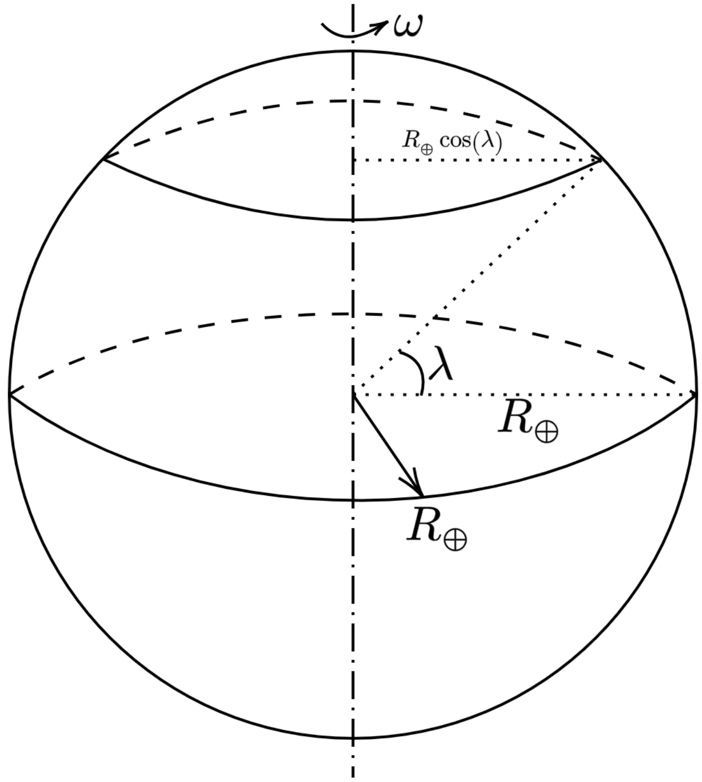

Figure 7:

We are given that $T = 24$ h and we know that most roughly $R_\oplus = 6400$ km, which you can estimate to be even 6000 km if you don't know what it is exactly (or when you need to perform quick calculations). At the equator, $r_{eq} = R_\oplus$, so

$$v_{eq} = \omega r_{eq} = \frac{2\pi}{T} R_\oplus = \frac{2 \times 3}{24 \times 3600 \text{ s}} \times 6.4 \times 10^6 \text{ m} \tag{31}$$

$$\boxed{v_{eq} = 440 \text{ m/s}} \tag{32}$$

Meantime,

$$a_{eq} = \frac{v_{eq}^2}{r_{eq}} = \omega^2 r_{eq} = \left(\frac{2\pi}{T}\right)^2 R_\oplus = \left(\frac{2 \times 3}{24 \times 3600 \text{ s}}\right)^2 \times 6.4 \times 10^3 \text{ m} \tag{33}$$

$$\boxed{a_{eq} = 0.031 \text{ m/s}^2 = (0.31\%) g} \tag{34}$$

The acceleration points to the rotation axis or to the center of the circle on which the person is.

At $\lambda = 37°$N, we have

$$r_\lambda = R_\oplus \cos(\lambda) \tag{35}$$

Thus, we get

$$v_\lambda = \omega r_\lambda = \omega R_\oplus \cos(\lambda) = \frac{2\pi}{T} R_\oplus \cos(\lambda) = \frac{2 \times 3}{24 \times 3600 \text{ s}} \times 6.4 \times 10^6 \text{ m} \times 0.6 \tag{36}$$

$$\boxed{v_\lambda = 270 \text{ m/s}} \tag{37}$$

As for the acceleration, we have

$$a_\lambda = \omega^2 r_\lambda = \left(\frac{2\pi}{T}\right)^2 R_\oplus \cos(\lambda) = \left(\frac{2 \times 3}{24 \times 3600 \text{ s}}\right)^2 \times 6.4 \times 10^6 \text{ m} \times 0.6 \tag{38}$$

$$\boxed{a_\lambda = 0.019 \text{ m/s}^2 = (0.19\%)g} \tag{39}$$

The acceleration in both cases points to the rotation axis or to the center of their respective circles; therefore, they are parallel.

v1 Oct 8, 2021