Adagio sostenuto

(a) A coin of mass m is attached to a cord of length R, which is fixed at one end. It moves in a horizontal circle with a constant angular speed ω on a frictionless tabletop. An identical second coin is on top of the first. Assume the top coin is just about the slip off the bottom one. Assume the dimensions of the coins are negligible compared to the cord length. Compute the tension in the cord and the coefficient of static friction, μ_1, between the coins in terms of m, R, ω, and g, where g is the gravitational acceleration.

(b) [From Physics 8.01, 2010 Fall, MIT] Assume the same configuration as in part (a) but remove the cord and replace the table with a turntable that has a coefficient of static friction μ_2 such that $\mu_2 > \mu_1$ and that turns at a constant angular speed Ω. Suppose the coins are not slipping. Compute the magnitude of the radial force exerted by the turntable on the bottom coin. As you increase the angular speed, which coin slips first or do they both slip at the same time? Find the highest angular speed Ω_{max} such that there is no slipping. Treat this part independently of part (a).

Solution • (a) The configuration is depicted in Fig. 1.

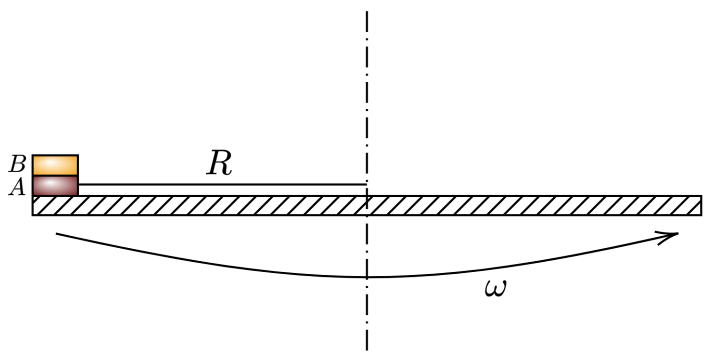


Figure 1:

I will call the bottom coin A and the top coin B. The free-body diagrams of the two coins are given in Fig. 2.

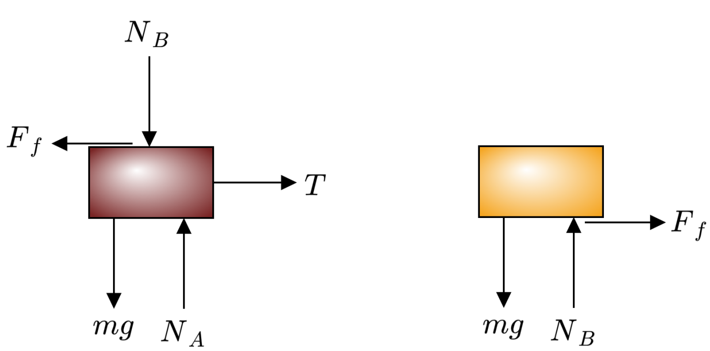


Figure 2:

A and B both have a centripetal acceleration $a_{cp} = \omega^2 R$ because they do uniform circular motion. The equation of motion is given by the Newton Second Law, $\mathbf{F}_{\text{net}} = m\mathbf{a}$. Let's consider the coins and the vertical and radial directions separately. For A, we have

$$\begin{cases} \text{vertical} : mg - N_A + N_B = 0 \\ \text{radial} : T - F_f = ma_{cp} \end{cases} \tag{1}$$

For B, we get to write

$$\begin{cases} \text{vertical} : mg - N_B = 0 \\ \text{radial} : F_f = ma_{cp} \end{cases} \tag{2}$$

Then, we see that

$$F_f = m\omega^2 R \tag{3}$$
$$T = F_f + m\omega^2 R \tag{4}$$
$$\boxed{T = 2m\omega^2 R} \tag{5}$$
$$N_B = mg \tag{6}$$
$$N_A = N_B + mg = 2mg \tag{7}$$
$$F_f = \mu_1 N_B = \mu_1 mg = m\omega^2 R \tag{8}$$
$$\boxed{\mu_1 = \frac{\omega^2 R}{g}} \tag{9}$$

(b) The configuration is depicted in Fig. 3.

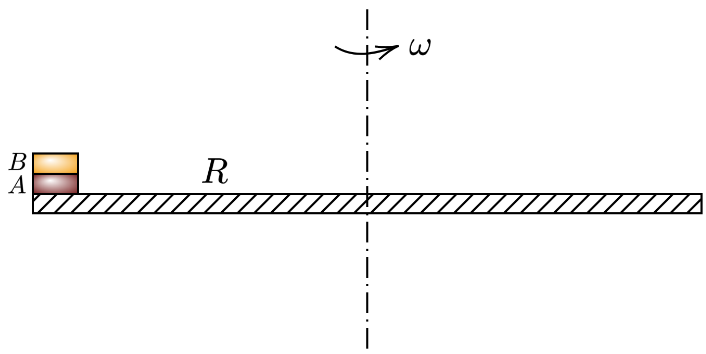


Figure 3:

The free-body diagrams are given in Fig. 4.

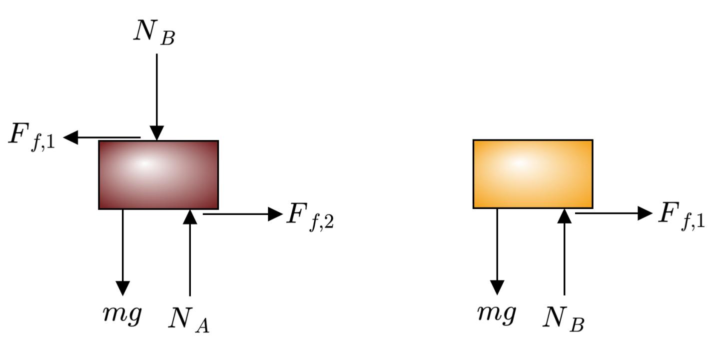


Figure 4:

We have a very similar set of equations of motion for both coins. For A, we have

$$\begin{cases} \text{vertical} : mg - N_A + N_B = 0 \\ \text{radial} : F_{f,2} - F_{f,1} = ma_{cp} \end{cases} \tag{10}$$

For B, we get

$$\begin{cases} \text{vertical} : mg - N_B = 0 \\ \text{radial} : F_{f,1} = ma_{cp} \end{cases} \tag{11}$$

Then, we see that

$$F_{f,1} = m\Omega^2 R \tag{12}$$
$$F_{f,2} = F_{f,1} + m\Omega^2 R \tag{13}$$
$$\boxed{F_{f,2} = 2m\Omega^2 R} \tag{14}$$

This is the radial force exerted by the turntable on the bottom coin to keep it in place, which turned out to be the same as the string tension in the previous part. This makes sense because friction does the job of the string now. As in part (a), we have

$$N_A = 2mg \tag{15}$$
$$N_B = mg \tag{16}$$

Now, suppose we have $\Omega = \Omega_{\max}$ so that things about to slip. Consider first the slipping condition of the bottom coin:

$$F_{f,2} = \mu_2 N_A = \mu_2(2mg) = 2m\Omega_{\max}^2 R \Rightarrow \mu_2 = \frac{\Omega_{\max}^2 R}{g} \tag{17}$$

Consider next the top coin:

$$F_{f,1} = \mu_1 N_B = \mu_1 mg = m\Omega_{\max}^2 R \Rightarrow \mu_1 = \frac{\Omega_{\max}^2 R}{g} \tag{18}$$

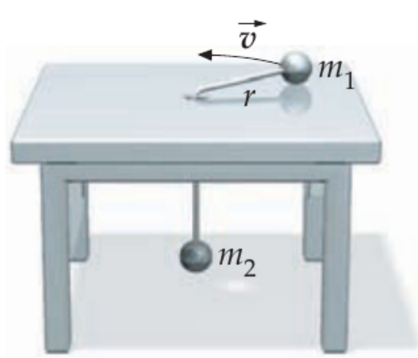
Since we have $\mu_2 > \mu_1$, the critical (or maximum) angular speed is given by

$$\boxed{\Omega_{\max} = \sqrt{\frac{\mu_1 g}{R}}} \tag{19}$$

and hence the top coin will slip off the bottom one first.

Allegretto (Problem 5-80 from Tipler's)

A small object of mass m_1 moves in a circular path of radius r on a frictionless horizontal tabletop as shown. It is attached to a string that passes through a small frictionless hole in the center of the table. A second object with a mass of m_2 is attached to the other end of the string. Derive an expression for r in terms of m_1, m_2, and the time P for one revolution.



Solution • Suppose I indicate the vertically upward (opposite to the gravity) direction by $+z$ and the radially outward direction by $+\rho$. The free-body diagrams are given in Fig. 5.

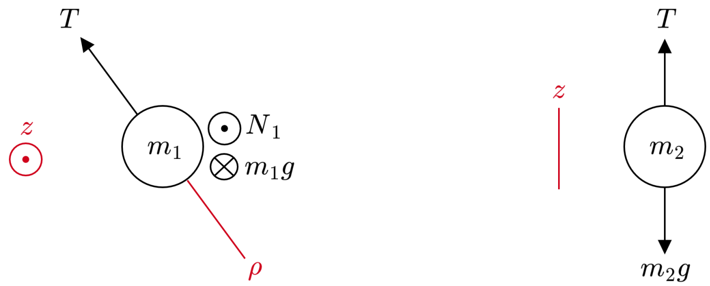


Figure 5:

m_1 will have centripetal acceleration, $a_{cp} = \omega^2 r$. m_2 is suspended midair so it has no net force on it. Using the Newton Second Law, $\mathbf{F}_{\text{net}} = m\mathbf{a}$, we see that the equation of radial motion for m_1 becomes

$$-T = -m_1\omega^2 r \Rightarrow T = m_1\omega^2 r \tag{20}$$

and that the equation of vertical motion for m_2 reads

$$T - m_2 g = 0 \Rightarrow T = m_2 g \tag{21}$$

Comparing the two expressions for T and using the relation $\omega = 2\pi/P$, we get

$$\boxed{r = \left(\frac{P}{2\pi}\right)^2 \frac{m_2 g}{m_1}} \tag{22}$$

Presto agitato (adapted from Brilliant.org)

Alice and Bob are throwing a ball to each other on a merry-go-round, which has a radius R and is rotating in the counterclockwise direction when viewed from top at a constant angular speed ω. Charlie is on the ground next to them. At $t = 0$, Alice is at the 6 o'clock position at the edge and throws the ball with a speed v to Bob, who is at the 3 o'clock position at the edge. Bob catches the ball at $t = \tau$, just after he completes a quarter turn. Assume the dimensions of the kids and the ball are negligible compared to the size of the merry-go-round. Assume v is large enough that the ball goes parallel to the ground as if there were no gravity.

(a) Describe mathematically the position of the ball in Charlie's reference frame.

(b) Describe mathematically the position of the ball in Alice and Bob's reference frame.

(c) Compute the average speed of the ball, i.e. $v' = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, |\mathbf{v}'|$, in Alice and Bob's reference frame. Is $v' < v$, $v' = v$, or $v' > v$? Why? Name the force(s) that cause this. (Hint: You may use Wolfram Alpha to directly compute the integral you get.)

Solution • (a) The configuration is depicted in Fig. 6.

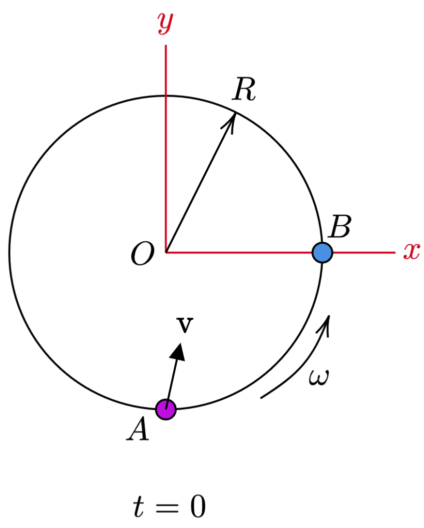


Figure 6:

Let Charlie put the origin of his reference frame at the center of the merry-go-round, point O. Then, we say that the indicated xy coordinate system belongs to an inertial frame. According to Charlie, the ball was initially at $\mathbf{x}_0 = -R\hat{\mathbf{y}}$. At time $t = \tau = P/4$, where P is the period of the merry-go-round, the ball lands on the point $\mathbf{x}(\tau) = R\hat{\mathbf{y}}$. Thus, according to Charlie, the ball moves in a straight line with constant velocity, $\mathbf{v} = v\hat{\mathbf{y}}$ where $v = (2R)/\tau = 8R/P$:

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}t = -R\hat{\mathbf{y}} + vt\hat{\mathbf{y}} \tag{23}$$

$$\boxed{\mathbf{x}(t) = (-R + vt)\hat{\mathbf{y}}} \tag{24}$$

(b) Let's go to the rotating coordinate system of Alice and Bob. Let's indicate the axes of this frame by x' and y'. Put Alice at $(x', y') = (R, 0)$ and Bob at $(x', y') = (0, R)$. Now, this frame rotates with ω so that we can write $\varphi = \varphi_0 + \omega t$ (see Fig. 7).

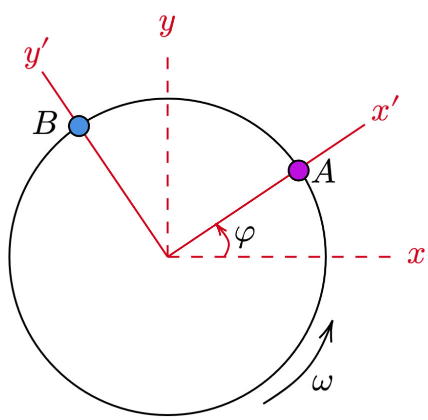


Figure 7:

We can relate the unit vectors $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ to $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}'\}$ as

$$\begin{cases} \hat{\mathbf{x}}' = \hat{\mathbf{x}}\cos(\varphi) + \hat{\mathbf{y}}\sin(\varphi) \\ \hat{\mathbf{y}}' = -\hat{\mathbf{x}}\sin(\varphi) + \hat{\mathbf{y}}\cos(\varphi) \end{cases} \tag{25}$$

with the inverse transformation given by

$$\begin{cases} \hat{\mathbf{x}} = \hat{\mathbf{x}}\cos(\varphi) - \hat{\mathbf{y}}\sin(\varphi) \\ \hat{\mathbf{y}} = \hat{\mathbf{x}}\sin(\varphi) + \hat{\mathbf{y}}\cos(\varphi) \end{cases} \tag{26}$$

Thus, we can write the position of the ball in the rotating frame as

$$\mathbf{x}(t) \to \mathbf{x}'(t) = (-R+vt)[\hat{\mathbf{x}}'\sin(\varphi) + \hat{\mathbf{y}}'\cos(\varphi)] = (-R+vt)[\hat{\mathbf{x}}'\sin(\varphi_0+\omega t) + \hat{\mathbf{y}}'\cos(\varphi_0+\omega t)] \tag{27}$$

At $t = 0$, Alice is at 6 o'clock, so let's take $\varphi_0 = -\frac{\pi}{2}$ for consistency. Then,

$$\boxed{\mathbf{x}'(t) = (-R+vt)[-\hat{\mathbf{x}}'\cos(\omega t) + \hat{\mathbf{y}}'\sin(\omega t)]} \tag{28}$$

(c) Now that we know where the ball is at all times, we can get the velocity by taking a time derivative:

$$\begin{aligned} \mathbf{v}' &= \dot{\mathbf{x}}' \\ &= v[-\hat{\mathbf{x}}'\cos(\omega t) + \hat{\mathbf{y}}'\sin(\omega t)] + (-R+vt)\omega[\hat{\mathbf{x}}'\sin(\omega t) + \hat{\mathbf{y}}'\cos(\omega t)] \\ &= \hat{\mathbf{x}}'[-v\cos(\omega t) + (-R+vt)\omega\sin(\omega t)] + \hat{\mathbf{y}}'[v\sin(\omega t) + (-R+vt)\omega\cos(\omega t)] \end{aligned} \tag{29}$$

Let's take the square. Though it seems like we have six terms, the cross terms will vanish and the other will go like $\cos(\omega t)^2 + \sin(\omega t)^2$, so we have

$$\mathbf{v}'^2 = \mathbf{v}' \cdot \mathbf{v}' = v^2 + (-R+vt)^2\omega^2 \tag{30}$$

Then, the average speed as defined in the problem statement is given by

$$\begin{aligned} v' &= \frac{1}{\tau}\int_0^\tau \mathrm{d}t\ \sqrt{\mathbf{v}'^2} \\ &= \frac{1}{\tau}\int_0^\tau \mathrm{d}t\ \sqrt{v^2 + (-R+vt)^2\omega^2} \\ &= \frac{v}{\tau}\int_0^\tau \mathrm{d}t\ \sqrt{1 + \frac{\omega^2}{v^2}(vt-R)^2} \\ &= \frac{v}{\tau}\int_0^\tau \mathrm{d}t\ \sqrt{1 + \omega^2\left(t - \frac{R}{v}\right)^2} \\ &= \frac{v}{\tau}\int_0^\tau \mathrm{d}t\ \sqrt{1 + \frac{4\pi^2}{P^2}\left(t - \frac{P}{8}\right)^2} \\ &= \frac{v}{\tau}\int_0^\tau \mathrm{d}t\ \sqrt{1 + 4\pi^2\left(\frac{t}{P} - \frac{1}{8}\right)^2} \end{aligned} \tag{31}$$

Let $u := \frac{t}{P} = \frac{t}{4\tau}$ so that $\mathrm{d}u = \frac{\mathrm{d}t}{4\tau}$ and $\int_{t=0}^{\tau} = \int_{u=0}^{1/4}$:

$$v' = 4v\int_0^{1/4} \mathrm{d}u\ \sqrt{1 + 4\pi^2\left(u - \frac{1}{8}\right)^2} \tag{32}$$

The analytical expression for this integral is cumbersome and does not bring anything new to our understanding of the problem. Let's compute it numerically using Wolfram Alpha. When I type `numerically integrate 4 sqrt(1 + 4 Pi^2 (x - 1/8)^2) from 0 to 1/4`, I get

$$v' \approx 1.0949v \tag{33}$$

which is 10% larger than v. This is due to the *centrifugal* (not centripetal) and Coriolis forces.

Let's see what's going on. In Fig. 8, I plot the path (indicated by the red line from A to B) the ball follows in both frames. It is clear that the second path is slightly longer. Since it takes the same amount of time for the ball to reach Bob, we see that the ball speed should be higher in the rotating frame than the outside, inertial frame.

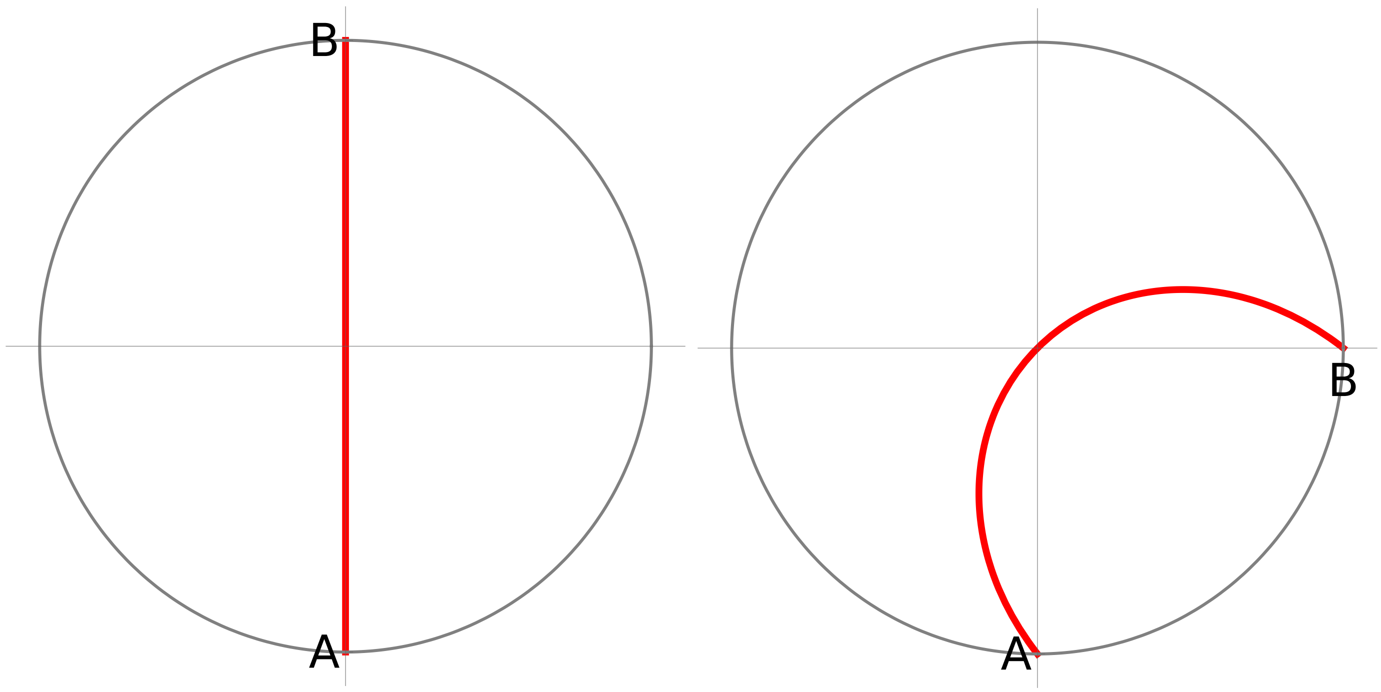


Figure 8: