Phys 507 Recitation Sessions

TA: Kağan Şimşek

Fall, 2017

November 9, 2017

Problem 1

Sakurai's, 1.4.c.

Problem 2

Sakurai's, 1.24.

Problem 3

Sakurai's, 1.33.

Problem 4[1]

Consider a quantum mechanical system which is described by a two dimensional Hilbert space spanned by basis vectors denoted $|1\rangle$ and $|2\rangle$. Let us introduce an operator A whose matrix elements in this basis are

$$\langle 1|A|1\rangle = \langle 2|A|2\rangle = a$$

$$\langle 1|A|2\rangle = \langle 2|A|1\rangle = b$$

(a) Find the eigenvectors and eigenvalues of A.

(b) Suppose the system is in the state

$$|\alpha\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{i}{\sqrt{2}}|2\rangle$$

What is the probability that when A is measured the result is a? b? $a+b$? $a-b$?

(c) Compute $\langle \Delta A^2 \rangle$ for this state.

Problem 5[2]

A brief review of Stern-Gerlach experiment.

November 16, 2017

Problem 6

Sakurai's, 1.28.

Problem 7

Sakurai's, 2.6.

Problem 8

Sakurai's, 2.23.

Problem 9

Sakurai's, 2.10.

November 23, 2017

Problem 10

Sakurai's, 2.16.

Problem 11

Sakurai's, 2.25.

Problem 12

Sakurai's, 2.28.a.

Problem 13

Sakurai's, 2.30.

December 7, 2017

Problem 14

Sakurai's, 2.22.

Problem 15

Sakurai's, 2.27.

Problem 16

Sakurai's, 2.32.

[1] Here, *Midterm 1, Problem 1.*

[2] Here, pp. 64-69.

1 Extra 1 [1]

Find the representation of the position operator in the momentum space. Solve the Schrödinger equation in the momentum space under the potential $V(x) = -eEx$.

2 Extra 2 : Sakurai 2.16, 17, 20, 39

3 Extra 3

The coherent states of the simple harmonic oscillator (SHO) are defined as the eigenkets of the annihilation operator, $a|\lambda\rangle = \lambda|\lambda\rangle$.

(a) Show that

$$|\lambda\rangle = \Delta(\lambda)|0\rangle \tag{3.1}$$

where $|0\rangle$ is the ground state of the SHO and the *displacement* operator, $\Delta(\lambda)$, is defined as

$$\Delta(\lambda) := e^{\lambda a^\dagger - \lambda^* a} \tag{3.2}$$

(b) Show that $Me^L M^{-1} = e^{MLM^{-1}}$ for any linear operators L and M. Compute $U_0(t)^\dagger \Delta(\lambda) U_0(t)$ where $U_0(t)$ is the usual time-evolution operator, $U_0(t) = e^{-iH_0 t/\hbar}$, and H_0 is the SHO Hamiltonian, $H_0 = P^2/2m + m\omega_0^2 X^2/2$. By using the result, obtain the state ket at a later time, $|\alpha, t_0 = 0; t\rangle$, assuming the state is initially in one of the coherent states, $|\alpha, t_0 = 0\rangle = |\lambda_0\rangle$.

(c) Now suppose that there appears a constant external force, f, which produces an interaction term, $H_1 = -fX$. Obtain the second-order ordinary differential equation that the position operator, X, satisfies.

(d) In quantum mechanics, in addition to the Schrödinger and Heisenberg pictures, there is another one, called the *Dirac* (or *interaction*) picture. In this framework, both states and operators are evolving in time.

We define an *intermediate* state ket, say α_I, via $|\alpha, t_0 = 0; t\rangle = e^{-iH_0 t/\hbar}|\alpha_I, t_0 = 0; t\rangle$. Noting that the total Hamiltonian now becomes $H = H_0 + H_1$, show that it satisfies the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\alpha_I, t_0 = 0; t\rangle = H_I(t)|\alpha_I, t_0 = 0; t\rangle \tag{3.3}$$

where $H_I(t) := e^{iH_0 t/\hbar} H_1 e^{-iH_0 t/\hbar}$. Since the interaction Hamiltonian, H_1, is a linear function of the position operator, X, we expect to have $H_I(t) = g(t)^* a + g(t) a^\dagger$. By using the Baker-Campbell-Hausdorff formula, obtain $g(t)$.

(e) From (3.3), we can deduce that there exists an *intermediate* time-evolution operator, $U_I(t)$, that satisfies

$$i\hbar \frac{\partial}{\partial t} U_I(t) = H_I(t) U_I(t) \tag{3.4}$$

By using the ansätz $U_I(t) = e^{h(t)a^\dagger - h(t)^* a} e^{i\beta(t)}$, derive the equations that $h(t)$ and $\beta(t)$ satisfy. Note that $\beta(t)$ is a real-valued function.

(f) The motivation in employing the Dirac picture is that we partition a given Hamiltonian so that we have the complete solutions to one part, and we treat the rest as a *perturbation*.

Assuming that the intermediate state ket is initially in one of the coherent states, $|\alpha_I, t_0 = 0\rangle = |\lambda_0\rangle$, first obtain $|\alpha_I, t_0 = 0; t\rangle$ by acting the *intermediate* time-evolution operator, $U_I(t)$, on this initial state. By using the result, find the final state ket by evolving it further with the usual time-evolution operator, $U_0(t)$. Demonstrate that the final state is of the form $|\alpha, t_0 = 0; t\rangle = e^{i\gamma(t)}|\lambda(t)\rangle$ where γ is some time-dependent phase. Express $\lambda(t)$ in terms of λ_0, $h(t)$, and any other parameters relevant to the problem.

(g) Obtain the function $h(t)$. Use it to explicitly compute $\lambda(t)$. Letting $x(t) := \sqrt{2\hbar/m\omega_0}\,\mathrm{Re}\,\lambda(t)$, discuss whether it solves the equation of motion for the position operator you obtained in part (b).

4 Extra 4 [2]

Consider the Hamiltonian

$$H = \frac{L^2}{2I}$$

Describe the eigenvalues and the eigenfunctions. Discuss the degeneracy. What happens if we modify the Hamiltonian into

$$H' = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_2}$$

5 Extra 5 : Sakurai 3.5, 14, 15, 17, 18

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6 Extra 6 [3]

Given $|R, j\rangle = D_z^j(R)|j, j\rangle$ where $D_z^j(R) = e^{iJ_3\varphi/\hbar}$, evaluate $J_3|R, j\rangle$. What are the Euler angles of the operator $D^j(R)$ that satisfies $D^j(R) J_3 D^j(R)^{-1} = \vec{J} \cdot \hat{n}$. Finally, show that $\vec{J} \cdot \hat{n}|R, j\rangle = j|R, j\rangle$.

[1] S. Kurkcuoglu, "Phys 507 Homework 2," Nov. 2017.

[2] S. Kurkcuoglu, "Phys 507 Homework 3," Dec. 2017.

[3] S. Kurkcuoglu, "Phys 507 Homework 3," Dec. 2017.

(1)

507 RECIT 1

Sakurai, 1.4.c

Consider the most general form: w/ $A = A^\dagger$ and $A|n\rangle = a_n|n\rangle$,

$$f(A) = \sum_j c_j A^j$$

$$A = \sum_n a_n |n\rangle\langle n|$$

$$A^2 = \sum_{nm} a_n a_m |n\rangle \underbrace{\langle n|m\rangle}_{\delta_{nm}} \langle m| = \sum_n a_n^2 |n\rangle\langle n|$$

...

$$A^j = \sum_n a_n^j |n\rangle\langle n|$$

$$\therefore f(A) = \sum_j c_j \sum_n a_n^j |n\rangle\langle n|$$

$$= \sum_n \left(\sum_j c_j a_n^j \right) |n\rangle\langle n|$$

$$= \sum_n f(a_n) |n\rangle\langle n|$$

$$\therefore e^{if(A)} = \sum_n e^{if(a_n)} |n\rangle\langle n|$$

e.g $e^{-iHt} = \sum_n e^{-iE_n t} |n\rangle\langle n|$ w/ $H|n\rangle = E_n|n\rangle$

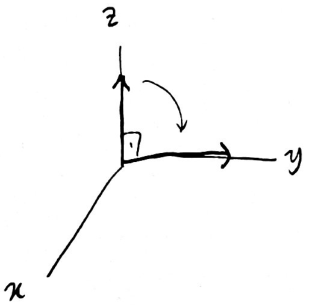
2

Sakurai 1.24

(a) $$A := \frac{1}{\sqrt{2}}(1 + i\sigma_x) = \frac{1}{\sqrt{2}}\left(1 + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

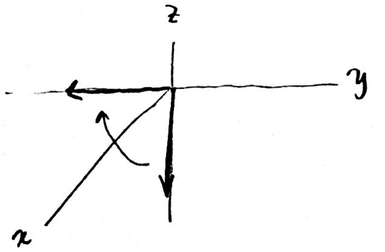
Let us apply it on something we know:

$$A|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = |y+\rangle$$



$$A|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$= \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = i|y-\rangle = \underbrace{e^{i\pi/2}}_{\text{phase} \therefore \text{ no phys. sig.}}|y-\rangle$$



$\therefore$ This is a rotator about the x by 90° cw.

(b) $$S_z \doteq \begin{pmatrix} \langle y+|S_z|y+\rangle & \langle y+|S_z|y-\rangle \\ \langle y-|S_z|y+\rangle & \langle y-|S_z|y-\rangle \end{pmatrix}$$

- Method 1: Use A above.

$|y+\rangle = A|+\rangle$

$|y-\rangle = -iA|-\rangle$

③

$$\langle y+|S_z|y+\rangle = \langle +|A^\dagger S_z A|+\rangle$$

$$\langle y+|S_z|y-\rangle = -i\langle +|A^\dagger S_z A|-\rangle$$

$$\langle y-|S_z|y+\rangle = i\langle -|A^\dagger S_z A|+\rangle$$

$$\langle y-|S_z|y-\rangle = \langle -|A^\dagger S_z A|-\rangle$$

$$A^\dagger S_z A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1-1 & 2i \\ -2i & 1-1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

$$\therefore \langle y+|S_z|y+\rangle = (1\ \ 0)\,\frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\langle y+|S_z|y-\rangle = -i\,(1\ \ 0)\,\frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}$$

$$\langle y-|S_z|y+\rangle = i\,(0\ \ 1)\,\frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}$$

$$\langle y-|S_z|y-\rangle = (0\ \ 1)\,\frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\therefore S_z \doteq \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = \frac{1}{2}\sigma_x$$

- Method 2

$$|y\pm\rangle = \frac{|+\rangle \pm i|-\rangle}{\sqrt{2}} \Rightarrow |+\rangle = \frac{|y+\rangle + |y-\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|y+\rangle - |y-\rangle}{\sqrt{2}\,i}$$

$$S_z = \frac{1}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) = \cdots = \frac{1}{2}\left(|y+\rangle\langle y-| + |y-\rangle\langle y+|\right)$$

Then evaluate the matrix elements.

④

Sakurai 1.33

(a) i. $\langle p|\mathbf{x}|\alpha\rangle = \langle p|\mathbf{x}\int dx\, |x\rangle\langle x|\alpha\rangle$

$$= \int dx\, \langle p|\mathbf{x}|x\rangle\langle x|\alpha\rangle$$

$$= \int dx\, x\langle p|x\rangle\langle x|\alpha\rangle$$

$$= \int dx\, dp'\; x\underbrace{\langle p|x\rangle}_{\frac{e^{-ipx}}{\sqrt{2\pi}}}\underbrace{\langle x|p'\rangle}_{\frac{e^{ip'x}}{\sqrt{2\pi}}}\langle p'|\alpha\rangle$$

$$= \frac{1}{2\pi}\int dx\, dp'\; x\, e^{i(p'-p)x}\;\langle p'|\alpha\rangle$$

$$= \frac{1}{2\pi}\int dx\, dp'\left(-\frac{1}{i}\frac{\partial}{\partial p}e^{i(p'-p)x}\right)\langle p'|\alpha\rangle$$

$$= -\frac{1}{i}\frac{\partial}{\partial p}\int dp'\,\Big(\underbrace{\int\frac{dx}{2\pi}e^{i(p'-p)x}}_{\delta(p'-p)}\Big)\langle p'|\alpha\rangle$$

$$= -\frac{1}{i}\frac{\partial}{\partial p}\langle p|\alpha\rangle \quad \text{qed}$$

$\therefore\ \mathbf{x} \to -\frac{1}{i}\frac{\partial}{\partial p}$ in momentum space

ii. $\langle\beta|\mathbf{x}|\alpha\rangle = \int dp\, dp'\; \langle\beta|p\rangle\underbrace{\langle p|\mathbf{x}|p'\rangle}_{-\frac{1}{i}\frac{\partial}{\partial p}\underbrace{\langle p|p'\rangle}_{\delta(p-p')}}\langle p'|\alpha\rangle$

$$= -\frac{1}{i}\int dp\, dp'\;\langle\beta|p\rangle\frac{\partial}{\partial p}\delta(p-p')\;\langle p'|\alpha\rangle$$

$$= -\frac{1}{i}\int dp\;\langle\beta|p\rangle\frac{\partial}{\partial p}\underbrace{\int dp'\,\delta(p-p')\langle p'|\alpha\rangle}_{\langle p|\alpha\rangle}$$

$$= -\frac{1}{i}\int dp \langle\beta|p\rangle \frac{\partial}{\partial p}\langle p|\alpha\rangle$$

$$= \int dp\, \psi_\beta(p)^* \frac{-1}{i}\frac{\partial}{\partial p}\psi_\alpha(p) \qquad \text{qed}$$

(6) $T(x) = e^{-iPx}$ translation in space, generated by linear momentum.

$\mathcal{U}(p) := e^{iXp}$ translation in momentum?

$$T(x) = \int dp\, e^{-ipx}|p\rangle\langle p|$$

$$T(a)|x\rangle = \int dp\, e^{-ipa}|p\rangle\langle p|x\rangle$$

$$= \int dx'dp\, e^{-ipa}\frac{e^{-ipx}}{\sqrt{2\pi}}|x'\rangle\langle x'|p\rangle$$

$$= \int dx'dp\, e^{-ipa}\frac{e^{-ipx}}{\sqrt{2\pi}}\frac{e^{ipx'}}{\sqrt{2\pi}}|x'\rangle$$

$$= \int dx'dp\, \frac{e^{ip(x'-(x+a))}}{2\pi}|x'\rangle$$

$$= \int dx' \Big(\underbrace{\int dp\, \frac{e^{ip(x'-(x+a))}}{2\pi}}_{\delta(x'-(x+a))}\Big)|x'\rangle$$

$$= |x+a\rangle$$

Simile,

$$\mathcal{U}(p) = \int dx\, e^{ixp}|x\rangle\langle x|$$

$$\mathcal{U}(k)|p\rangle = \int dx\, e^{ikx}|x\rangle\langle x|p\rangle$$

$$= \int dx\, dp'\, e^{ikx}|p'\rangle\langle p'|x\rangle\langle x|p\rangle$$

$$= \int dx\, dp'\, e^{ikx}|p'\rangle\frac{e^{-ip'x}}{\sqrt{2\pi}}\frac{e^{ipx}}{\sqrt{2\pi}}$$

⑥

$$= \int dp' \left(\int dx \, \frac{e^{ix(p+k-p')}}{2\pi} \right) |p'\rangle$$

$$= \int dp' \, \delta(p' - (p+k)) \, |p'\rangle$$

$$= |p+k\rangle \quad \checkmark$$

So it is indeed the translation operator in the momentum space.

Problem 4

$$\langle 1|A|1\rangle = \langle 2|A|2\rangle = a \quad \Rightarrow \quad A \supset a|1\rangle\langle 1| + a|2\rangle\langle 2|$$

$$\langle 1|A|2\rangle = \langle 2|A|1\rangle = b \quad \Rightarrow \quad A \supset b|1\rangle\langle 2| + b|2\rangle\langle 1|$$

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

(a) $|A - \lambda| = 0$

$$\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = 0 \quad \Rightarrow \quad \boxed{\lambda = a \pm b}$$

$$(A - \lambda)|\psi\rangle = 0$$

$$|\psi\rangle \doteq \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(a-\lambda)c_1 + b c_2 = 0$$

$$c_2 = \frac{\lambda - a}{b} c_1 = \frac{a \pm b - a}{b} c_1 = \pm c_1 \qquad \therefore \boxed{|\psi_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}}$$

(7)

(b) $|\alpha\rangle = \frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$

$\boxed{P(a) = 0}$ ∵ no such value in the spectrum of A.

$\boxed{P(b) = 0}$ simile

To see this,

$$A|\psi'\rangle = a|\psi'\rangle$$

If $|\psi'\rangle$ exists, we should be able to construct it:

$$|\psi'\rangle = \gamma_1|1\rangle + \gamma_2|2\rangle$$

$$A|\psi'\rangle = \gamma_1(a+b) + \gamma_2(a-b)|2\rangle = a\gamma_1|1\rangle + a\gamma_2|2\rangle$$

Since $|1\rangle$ and $|2\rangle$ are lin. indep,

$$\gamma_1(a+b) = a\gamma_1 \Rightarrow \gamma_1 = 0$$

$$\gamma_2(a-b) = a\gamma_2 \Rightarrow \gamma_2 = 0$$

So $|\psi'\rangle$ is a trivial state:

$$|\psi'\rangle = 0$$

Then

$$P(a) = |\langle\psi'|\alpha\rangle|^2 = 0 \quad \text{trivially.}$$

$$P(a\pm b) = |\langle\psi_\pm|\alpha\rangle|^2$$

$$= \left|\frac{\langle 1| \pm \langle 2|}{\sqrt{2}} \; \frac{|1\rangle + i|2\rangle}{\sqrt{2}}\right|^2$$

$$= \frac{1}{4}|1 \pm i|^2$$

$$\boxed{P(a\pm b) = \frac{1}{2}}$$

8

(c) $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$|\alpha\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\langle A \rangle = \langle \alpha | A | \alpha \rangle = \frac{1}{\sqrt{2}} (1 \quad -i) \begin{pmatrix} a & b \\ b & a \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = a$

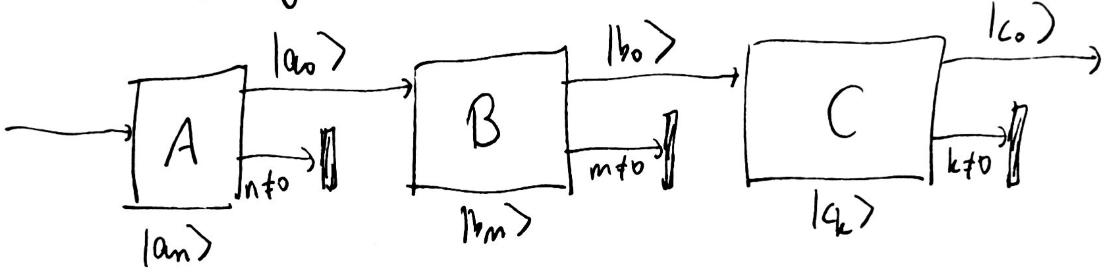
$\langle A^2 \rangle = \langle \alpha | A^2 | \alpha \rangle = \frac{1}{\sqrt{2}} (1 \quad -i) \begin{pmatrix} a & b \\ b & a \end{pmatrix}^2 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = a^2 + b^2$

$\therefore \langle \Delta A^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 = a^2 + b^2 - a^2$

$\boxed{\langle \Delta A^2 \rangle = b^2}$

Extra 1

Philosophy of Measurement in QM



T : transition amplitude (or probability amp.)

$$T = \langle \psi_f | \underbrace{Q}_{\text{any observable}} | \psi_i \rangle$$

Case 1 Measure and record b_0 only:

$$T = \underbrace{\langle c_0}_{\text{final}} \underbrace{| b_0 \rangle\langle b_0 |}_{\text{we force the B measurement to return } b_0 \text{ only}} \underbrace{a_0 \rangle}_{\text{initial}}$$

$\therefore$ we 'project' a_0 on b_0.

In terms of Feynman's notation:

$$\underbrace{\langle \psi_f}_{\text{final state}} | \underbrace{\text{operation}}_{\text{a sequence of observations/operations}} | \underbrace{\psi_i \rangle}_{\text{initial state}}$$

Extra2

Here, in this case,

$$\text{operation} = \underbrace{\Lambda^B_0}_{\text{projection operator for B kets}} = |b_0\rangle\langle b_0|$$

So probability $= |T|^2$

$$= |\langle c_0|b_0\rangle\langle b_0|a_0\rangle|^2$$

$$= |\langle c_0|b_0\rangle|^2 \, |\langle b_0|a_0\rangle|^2$$

Case 2 Measure and record all possible b_0's.

$$|T|^2 = \sum_{b_0} |\langle c_0|b_0\rangle|^2 \, |\langle b_0|a_0\rangle|^2$$

Notice : We do not start from the amplitude :

$$T = \sum_{b_0} \langle c_0|b_0\rangle\langle b_0|a_0\rangle$$

$\because$ this ~~[illegible]~~ would mean that the 'transition' from a_0 to c_0 is indeed over all b_0's —— that's not the case here. But we want the total ~~[illegible]~~

Extra 3

'probability' if we consider all possible 'paths'. Recall we measure only the probability. So

$$|T|^2 = \sum_{b_0} |\langle c_0 | b_0 \rangle|^2 \, |\langle b_0 | a_0 \rangle|^2 \qquad (*)$$

is indeed the result.

- Case 3 Do not measure or record any information coming out of B apparatus.

Now we have the following:

$$\begin{aligned} T &= \langle c_0 | b_0 \rangle \langle b_0 | a_0 \rangle \\ &+ \langle c_0 | b_1 \rangle \langle b_1 | a_0 \rangle \\ &+ \cdots \\ &+ \langle c_0 | b_\infty \rangle \langle b_\infty | a_0 \rangle \\ &= \sum_{b_0} \langle c_0 | b_0 \rangle \langle b_0 | a_0 \rangle \end{aligned}$$

So that's why we sum over the intermediate states at the beginning.

extra 4.

Probability here is

$$|T|^2 = \Big|\sum_{b_0} \langle c_0|b_0\rangle\langle b_0|a_0\rangle\Big|^2$$

$$= \sum_{b_0 b_0'} \langle c_0|b_0\rangle\langle b_0|a_0\rangle \langle a_0|b_0'\rangle\langle b_0'|c_0\rangle$$

which is most definitely not equal to $(*)$.

Quote of the week:

"When in doubt, expand in a power series."

— Fermi

1

Natural units : $\hbar = c = 1$

507 RECIT 2

Sakurai 1.28

(a) $$[x, F(p)]_{cl} = \frac{\partial x}{\partial x}\frac{\partial F(p)}{\partial p} - \underbrace{\cancel{\frac{\partial x}{\partial p}}}_{0}\frac{\partial F(p)}{\partial x}$$

$$= \frac{\partial F(p)}{\partial p}$$

(b) $$[X, e^{iPa}] = ?$$

- Method 1 : Refer to the previous notes to see that

$$X \rightarrow -\frac{1}{i}\frac{\partial}{\partial p}$$

in momentum space :

$$[X, e^{iPa}] = X e^{iPa} - e^{iPa} X$$

$$= -\frac{1}{i}\frac{\partial}{\partial p}(e^{iPa}) - e^{iPa}\,\frac{-1}{i}\frac{\partial}{\partial p}$$

$$= -\frac{1}{i}\frac{\partial e^{iPa}}{\partial p} - \cancel{\frac{1}{i}e^{iPa}\frac{\partial}{\partial p}} + \cancel{\frac{1}{i}e^{iPa}\frac{\partial}{\partial p}}$$

$$= i\,\frac{\partial e^{iPa}}{\partial p}$$

$$= -a e^{iPa}$$

②

- Method 2: Expand the exponential in a power series.

$$[X, e^{iPa}] = [X, \sum_j c_j P^j]$$
$$= \sum_j c_j [X, P^j]$$

$$[X, P] = i$$
$$[X, P^2] = P[X,P] + [X,P]P = 2iP$$
$$[X, P^3] = P[X,P^2] + [X,P]P^2 = 3iP^2$$

$$\dots$$

$$[X, P^j] = j i P^{j-1} = i \frac{\partial P^j}{\partial P}$$

$$\therefore [X, e^{iPa}] = \sum_j c_j \; i \frac{\partial P^j}{\partial P}$$
$$= i \frac{\partial}{\partial P} \sum_j c_j P^j$$
$$= i \frac{\partial}{\partial P} e^{iPa}$$
$$= -a e^{iPa}$$

③

(c) $$X\left(e^{iPa}|x\rangle\right) = \left(e^{iPa}X + \underbrace{[X, e^{iPa}]}_{-ae^{iPa}}\right)|x\rangle$$

$$= e^{iPa}X|x\rangle - a e^{iPa}|x\rangle$$

$$= (x-a)\left(e^{iPa}|x\rangle\right) \qquad \text{qed}$$

Solusi 2.6

$$H = \frac{P^2}{2m} + V(X)$$

$$[H, X] = \left[\frac{P^2}{2m} + V(X), X \right]$$

$=0$

$$= \frac{1}{2m} [P^2, X]$$

$$= \frac{1}{2m} \left(P\underbrace{[P, X]}_{-i} + \underbrace{[P, X]}_{-i}P \right)$$

$$= -\frac{iP}{m}$$

$$[[H, X], X] = \left[-\frac{iP}{m}, X \right]$$

$$= -\frac{i}{m} \underbrace{[P, X]}_{-i}$$

$$= -\frac{1}{m}$$

$$[X, [H, X]] = \frac{1}{m}$$

(5)

$$\langle n | [X, [H, X]] | n\rangle = \langle n | [X, HX - XH] | n\rangle$$

$$= \langle n | XHX - X^2 H - HX^2 + XHX | n\rangle$$

$$= 2\langle n | XHX | n\rangle - \langle n | X^2 \underbrace{H | n\rangle}_{E_n} - \underbrace{\langle n | H}_{E_n} X^2 | n\rangle$$

$$= 2\left(\langle n | XHX | n\rangle - \langle n | X^2 | n\rangle E_n\right) \quad (=)$$

$$\langle n | XHX | n\rangle = \langle n | X \sum_m |m\rangle\langle m | H \sum_k |k\rangle\langle k | X | n\rangle$$

$$= \sum_{mk} \langle n | X | m\rangle \overbrace{\langle m | H | k\rangle}^{\delta_{mk}} \langle k | X | n\rangle$$

E_k

$$= \sum_{mk} \langle n | X | m\rangle E_k \delta_{mk} \langle k | X | n\rangle$$

$$= \sum_m \langle n | X | m\rangle\langle m | X | n\rangle E_m$$

$$= \sum_m |\langle n | X | m\rangle|^2 E_m$$

$$(=)\ 2\left(\sum_m |\langle n | X | m\rangle|^2 E_m - \underbrace{\langle n | X^2 | n\rangle}_{} E_n\right)$$

$$= \langle n | XX | n\rangle$$

$$= \langle n | X \sum_m |m\rangle\langle m | X | n\rangle$$

$$= \sum_m \langle n | X | m\rangle\langle m | X | n\rangle$$

$$= \sum_m |\langle n | X | m\rangle|^2$$

⑥

$$= 2\left(\sum_m |\langle n|X|m\rangle|^2 E_m - \sum_m |\langle n|X|m\rangle|^2 E_n \right)$$

$$= 2\sum_m |\langle n|X|m\rangle|^2 (E_m - E_n)$$

$$\langle n|[X,[H,X]]|n\rangle = \langle n|\frac{1}{m}|n\rangle = \frac{1}{m}\underbrace{\langle n|n\rangle}_{1} = \frac{1}{m}$$

$$\therefore 2\sum_m |\langle n|X|m\rangle|^2 (E_m - E_n) = \frac{1}{m}$$

$$\therefore \sum_m |\langle n|X|m\rangle|^2 (E_m - E_n) = \frac{1}{2m} \qquad \text{qed}$$

⑦

Sakurai 2.23

$$V(x) = \begin{cases} 0, & 0<x<L \\ \infty, & \text{o.w} \end{cases}$$ "particle in a box"

$$H|n\rangle = E_n|n\rangle$$

$$\langle x|n\rangle = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \quad n\in\mathbb{Z}^+$$ from elementary quantum mech.

$$|\alpha(0)\rangle = \sum_{n\geq 1} |n\rangle\langle n|\alpha(0)\rangle$$

$$\langle x|\alpha(0)\rangle = \delta\left(x-\frac{L}{2}\right)$$

$$\langle x|\alpha(t)\rangle = ?$$

$$|\alpha(t)\rangle = e^{-iHt}|\alpha(0)\rangle, \quad f(A) = \sum_n f(a_n)|a_n\rangle\langle a_n|$$

$$= \sum_{n\geq 1} e^{-iE_nt}|n\rangle\langle n|\alpha(0)\rangle$$

$$= \sum_{n\geq 1} e^{-iE_nt}|n\rangle\langle n|\int_0^L dx\,|x\rangle\langle x|\alpha(0)\rangle$$

$$= \sum_{n\geq 1} e^{-iE_nt}|n\rangle\int_0^L dx\,\underbrace{\langle n|x\rangle}_{\substack{=\langle x|n\rangle^* \\ =\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}}}\underbrace{\langle x|\alpha(0)\rangle}_{\delta(x-\frac{L}{2})}$$

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$$= \sum_{n \geq 1} e^{-iE_n t} |n\rangle \int_0^L dx \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \, \delta\left(x - \frac{L}{2}\right)$$

$$= \sum_{n \geq 1} e^{-iE_n t} |n\rangle \sqrt{\frac{2}{L}} \sin \frac{n\pi}{2}$$

$$\langle x | \alpha(t) \rangle = \sum_{n \geq 1} e^{-iE_n t} \underbrace{\langle x | n \rangle}_{\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}} \sqrt{\frac{2}{L}} \sin \frac{n\pi}{2}$$

$$= \sum_{n \geq 1} e^{-iE_n t} \frac{2}{L} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L}$$

⑨

Soluzioni 2. 10

$$H = \Delta \left(|L\rangle\langle R| + |R\rangle\langle L| \right)$$

(a) $|L\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |R\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\therefore H = \Delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|H - \lambda| = 0$$

$$\begin{vmatrix} -\lambda & \Delta \\ \Delta & -\lambda \end{vmatrix} = \lambda^2 - \Delta^2 = 0 \Rightarrow \lambda = \pm \Delta$$

$$(H - \lambda)|\pm\rangle = 0, \quad |\pm\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & \Delta \\ \Delta & -\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\lambda a + \Delta b = 0$$

$$b = \frac{\lambda}{\Delta} a = \begin{cases} a \\ -a \end{cases} = \pm a$$

$$\therefore |\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{|L\rangle \pm |R\rangle}{\sqrt{2}}$$

(10)

(b) $|\alpha(0)\rangle = |R\rangle\langle R|\alpha(0)\rangle + |L\rangle\langle L|\alpha(0)\rangle$

$=: |R\rangle c_R(0) + |L\rangle c_L(0)$

$$|\alpha(t)\rangle = e^{-iHt}|\alpha(0)\rangle$$

$$= e^{-iHt}\Big(c_R(0)\underbrace{|R\rangle}_{\frac{|+\rangle-|-\rangle}{\sqrt{2}}} + c_L(0)\underbrace{|L\rangle}_{\frac{|+\rangle+|-\rangle}{\sqrt{2}}}\Big)$$

$$= \frac{c_R(0)}{\sqrt{2}}\left(e^{-iHt}|+\rangle - e^{-iHt}|-\rangle\right) + \frac{c_L(0)}{\sqrt{2}}\left(e^{-iHt}|+\rangle + e^{-iHt}|-\rangle\right)$$

$$= \frac{c_R(0)}{\sqrt{2}}\left(e^{-i\Delta t}|+\rangle - e^{+i\Delta t}|-\rangle\right) + \frac{c_L(0)}{\sqrt{2}}\left(e^{-i\Delta t}|+\rangle + e^{+i\Delta t}|-\rangle\right)$$

$$= |+\rangle \frac{c_R(0)+c_L(0)}{\sqrt{2}} e^{-i\Delta t} + |-\rangle \frac{c_L(0)-c_R(0)}{\sqrt{2}} e^{+i\Delta t}$$

$$= \frac{|L\rangle+|R\rangle}{\sqrt{2}} \frac{c_R(0)+c_L(0)}{\sqrt{2}} e^{-i\Delta t} + \frac{|L\rangle-|R\rangle}{\sqrt{2}} \frac{c_L(0)-c_R(0)}{\sqrt{2}} e^{i\Delta t}$$

$$= |L\rangle\left(-i\sin\Delta t\; c_R(0) + \cos\Delta t\; c_L(0)\right)$$

$$+ |R\rangle\left(\cos\Delta t\; c_R(0) - i\sin\Delta t\; c_L(0)\right)$$

(11)

$$|\alpha(t)\rangle = |L\rangle\left(-i\sin\Delta t\, c_R(0) + \cos\Delta t\, c_L(0)\right)$$
$$+ |R\rangle\left(\cos\Delta t\, c_R(0) - i\sin\Delta t\, c_L(0)\right)$$

(c) $|\alpha(0)\rangle = |R\rangle \Rightarrow c_L(0) = 0,\ c_R(0) = 1$

$$\therefore |\alpha(t)\rangle = -i\sin\Delta t\, |L\rangle + \cos\Delta t\, |R\rangle$$

$$P(L) = |\langle L|\alpha(t)\rangle|^2$$
$$= \sin^2\Delta t$$

(d) $$\Psi = \begin{pmatrix} \langle L|\alpha(t)\rangle \\ \langle R|\alpha(t)\rangle \end{pmatrix} =: \begin{pmatrix} c_L(t) \\ c_R(t) \end{pmatrix}$$

$$= c_L(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_R(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= c_L(t)|L\rangle + c_R(t)|R\rangle$$

$$i\frac{\partial\Psi}{\partial t} = H\Psi$$

Assume $\exists\ \mathcal{U}$ s.t $\mathcal{U}^\dagger\mathcal{U} = 1$ et $\mathcal{U}^\dagger H \mathcal{U} = \begin{pmatrix} \Delta & \\ & -\Delta \end{pmatrix}$.
From elementary algebra, one such matrix is

$$\mathcal{U} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \sim|+\rangle & \sim|-\rangle \end{array}$$

(12)

$$i\frac{\partial\psi}{\partial t} = H\psi = H\mathcal{U}\mathcal{U}^{\dagger}\psi \quad \Big|\ \mathcal{U}^{\dagger}\rightarrow$$

$$i\frac{\partial}{\partial t}\underbrace{\mathcal{U}^{\dagger}\psi}_{\tilde{\psi}} = \underbrace{\underbrace{\mathcal{U}^{\dagger}H\mathcal{U}}_{\begin{pmatrix}\Delta & \\ & -\Delta\end{pmatrix}}}_{=:E}\underbrace{\mathcal{U}^{\dagger}\psi}_{\tilde{\psi}}$$

$$i\frac{\partial\tilde{\psi}}{\partial t} = E\tilde{\psi}$$

$$\frac{\partial\tilde{\psi}}{\partial t} = -iE\tilde{\psi}$$

$$\frac{\partial\tilde{\psi}}{\tilde{\psi}} = -iE\,\partial t$$

$$\ln\tilde{\psi}(t) = -iEt + \ln\tilde{\psi}(0)$$

$$\tilde{\psi}(t) = \tilde{\psi}(0)\,e^{-iEt} \qquad (*)$$

$$\tilde{\psi} = \mathcal{U}^{\dagger}\psi = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}c_L(t)\\ c_R(t)\end{pmatrix}$$

$$= \begin{pmatrix}\dfrac{c_L(t)+c_R(t)}{\sqrt{2}}\\[2ex] \dfrac{c_L(t)-c_R(t)}{\sqrt{2}}\end{pmatrix}$$

(13)

$$(*)\begin{cases} \dfrac{c_L(t)+c_R(t)}{\sqrt{2}} = e^{-i\Delta t}\,\dfrac{c_L(0)+c_R(0)}{\sqrt{2}} \\[2ex] \dfrac{c_L(t)-c_R(t)}{\sqrt{2}} = e^{+i\Delta t}\,\dfrac{c_L(0)-c_R(0)}{\sqrt{2}} \end{cases}$$

$$c_L(t) = \frac{1}{2}\left(e^{-i\Delta t}\,(c_L(0)+c_R(0)) + e^{i\Delta t}\,(c_L(0)-c_R(0))\right)$$

$$= c_L(0)\cos\Delta t - i c_R(0)\sin\Delta t$$

$$c_R(t) = \frac{1}{2}\left(e^{-i\Delta t}\,(c_L(0)+c_R(0)) - e^{i\Delta t}\,(c_L(0)-c_R(0))\right)$$

$$= -i c_L(0)\sin\Delta t + c_R(0)\cos\Delta t$$

$$\therefore \Psi = c_L(t)\,|L\rangle + c_R(t)\,|R\rangle$$

$$= \left(c_L(0)\cos\Delta t - i c_R(0)\sin\Delta t\right)|L\rangle$$

$$+\left(-i c_L(0)\sin\Delta t + c_R(0)\cos\Delta t\right)|R\rangle$$

which is the same as in part (b).

(14)

(e) $\tilde{H} = \Delta |L\rangle\langle R|$

$$\langle \alpha(t)|\alpha(t)\rangle \stackrel{?}{=} \langle \alpha(0)|\alpha(0)\rangle$$

$$\langle \alpha(0)|\underbrace{e^{i\tilde{H}^\dagger t}\, e^{-i\tilde{H}t}}_{e^{i(\tilde{H}^\dagger - \tilde{H})t}\ \neq 1 \ \because\ \tilde{H}^\dagger \neq \tilde{H}}|\alpha(0)\rangle \stackrel{?}{=} \langle \alpha(0)|\alpha(0)\rangle$$

$$\therefore\ \langle \alpha(t)|\alpha(t)\rangle \neq \langle \alpha(0)|\alpha(0)\rangle$$

Quote of the week:

"Whatever is not expressly forbidden is mandatory."

— Feynman

1

507 RECIT 3

Sakurai, 2.16

$$C_n(t) := \langle x(t) | x(0) \rangle$$

$$x(t) = \mathcal{U}^\dagger x(0) \mathcal{U}$$

$$= e^{iHt} x e^{-iHt}$$

$$C_n(t) = \langle n | e^{iHt} x e^{-iHt} x | n \rangle$$

$$a_\pm := \frac{1}{\sqrt{2m\omega\hbar}} (m\omega x \mp ip) \qquad (a \leftrightarrow a_-, \quad a^\dagger \leftrightarrow a_+)$$

$$a_+ |n\rangle = \sqrt{n+1}\, |n+1\rangle, \quad a_- |n\rangle = \sqrt{n}\, |n\rangle$$

$$\sqrt{2m\omega\hbar}\, a_+ = m\omega x - ip$$

$$\sqrt{2m\omega\hbar}\, a_- = m\omega x + ip$$

$$x = \frac{1}{2m\omega} \left(\sqrt{2m\omega\hbar}\, a_+ + \sqrt{2m\omega\hbar}\, a_- \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-)$$

$$p = \frac{1}{2i} \left(-\sqrt{2m\omega\hbar}\, a_+ + \sqrt{2m\omega\hbar}\, a_- \right)$$

$$= i\sqrt{\frac{m\omega\hbar}{2}} (a_+ - a_-)$$

(2)

$$C_n(t) = \langle n| e^{iHt} x e^{-iHt} x |n\rangle$$

$$= \langle n| e^{iHt} x e^{-iHt} \sum_m |m\rangle\langle m|x|n\rangle$$

$$= \sum_m \langle n|\underbrace{e^{iHt}}_{e^{iE_n t}} x \underbrace{e^{-iHt}|m\rangle}_{e^{-iE_m t}} \langle m|x|n\rangle$$

$$= \sum_m e^{i(E_n - E_m)t} \cdot \langle n|x|m\rangle\langle m|x|n\rangle$$

$$= \sum_m e^{i(E_n - E_m)t} \, |\langle n|x|m\rangle|^2$$

$$\langle n|x|m\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle n|a_+ + a_-|m\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\langle n|a_+|m\rangle + \langle n|a_-|m\rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\langle n|\sqrt{m+1}\,|m+1\rangle + \langle n|\sqrt{m}\,|m-1\rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{m+1}\, \delta_{n,m+1} + \sqrt{m}\, \delta_{n,m-1} \right)$$

③

$$|\langle n|x|m\rangle|^2 = \frac{\hbar}{2m\omega}\left(\sqrt{m+1}\,\delta_{n,m+1} + \sqrt{m}\,\delta_{n,m-1}\right)^2$$

$n=0$:

$$C_0(t) = \sum_m e^{i(E_0-E_m)t}\frac{\hbar}{2m\omega}\left(\sqrt{m+1}\,\delta_{0,m+1} + \sqrt{m}\,\delta_{0,m-1}\right)^2$$

$$= \frac{\hbar}{2m\omega}\Big(e^{i(E_0-E_0)t}(\overbrace{\delta_{01}+0}^{0})^2 + e^{i(E_0-E_1)t}(\underbrace{\sqrt{2}\,\delta_{02}}_{0}+\underbrace{\delta_{00}}_{1})^2$$

$$\underbrace{+\, e^{i(E_0-E_2)t}(\sqrt{3}\,\delta_{03} + \sqrt{2}\,\delta_{01})^2 + \dots}_{0}\Big)$$

$$= \frac{\hbar}{2m\omega} e^{iE_0-E_1t}$$

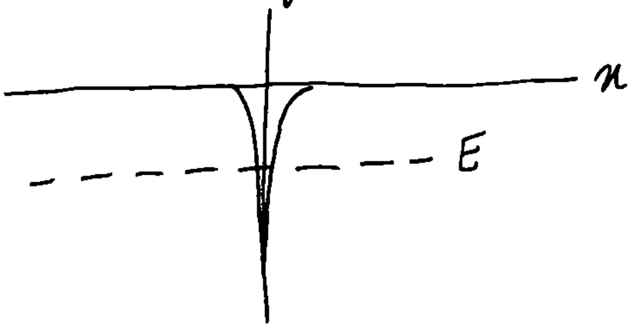
$$= \frac{\hbar}{2m\omega} e^{i\left(\frac{\hbar\omega}{2}-\frac{3\hbar\omega}{2}\right)t/\hbar}$$

$$\boxed{C_0(t) = \frac{\hbar}{2m\omega}e^{-i\omega t}}$$

④

Sakurai, 2.25

$V(x) = -\lambda\delta(x),\ \lambda > 0,\ t < 0,$ bound states



$$H\phi_n(x) = E_n\phi_n(x)$$

$$-\frac{\hbar^2}{2m}\phi_n'' - \lambda\delta(x)\phi_n = E_n\phi_n = -|E_n|\phi_n$$

$x > 0$:

$$-\frac{\hbar^2}{2m}\phi_n'' = -|E_n|\phi_n$$

$$\phi_n'' = \frac{2m|E_n|}{\hbar^2}\phi_n =: K^2\phi_n$$

$$\phi_n(x) = Ae^{-Kx} + \cancel{Be^{Kx}}$$

$x < 0$: (symmetric)

$$\phi_n(x) = Ce^{Kx}$$

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$$\phi_n(0^-) = \phi_n(0^+) \quad \Rightarrow \quad A = C$$

$x = 0$:

$$-\frac{\hbar^2}{2m}\phi_n'' - \lambda\delta(x)\phi_n = E_n\phi_n \quad \Big| \int_{0^-}^{0^+} dx$$

$$-\frac{\hbar^2}{2m}\left(\phi_n'(0^+) - \phi_n'(0^-)\right) - \lambda\phi_n(0) = 0$$

$$\phi_n'(0^+) - \phi_n'(0^-) = -\frac{2m\lambda}{\hbar^2}\phi_n(0)$$

$$-KA - KA = -\frac{2m\lambda}{\hbar^2}A$$

$$K = \frac{m\lambda}{\hbar^2}$$

$$K^2 = \frac{2m|E_n|}{\hbar^2} = \frac{m^2\lambda^2}{\hbar^4}$$

$$|E_n| = \frac{m\lambda^2}{2\hbar^2} \qquad \text{only one state}$$

$$E = -\frac{m\lambda^2}{2\hbar^2}$$

⑥

$$\psi(x) = \begin{cases} Ae^{-Kx}, & x > 0 \\ Ae^{Kx}, & x < 0 \end{cases}$$

$$= Ae^{-K|x|}$$

$$\int_{-\infty}^{\infty} dx \, |\psi(x)|^2 = 1$$

$$A^2 \int_{-\infty}^{\infty} dx \, e^{-2K|x|} = 1$$

$$2A^2 \int_0^{\infty} dx \, e^{-2Kx} = 1$$

$$2A^2 \frac{1}{2K} = 1$$

$$A = \sqrt{K} = \sqrt{\frac{m\lambda}{\hbar^2}}$$

$$\psi(x) = \sqrt{K} \, e^{-K|x|}, \quad t < 0$$

$$\therefore \langle x|\alpha(0)\rangle = \sqrt{K} \, e^{-K|x|}$$

⑦

$$\langle x|\alpha(t)\rangle = ?$$

$$H = \frac{P^2}{2m}, \quad t > 0$$

$$\langle x|\alpha(t)\rangle = \langle x|e^{-iHt/\hbar}|\alpha(0)\rangle$$

$$= \langle x|e^{-iP^2t/2m\hbar}|\alpha(0)\rangle$$

$$= \langle x|e^{-iP^2t/2m\hbar}\int dp\,|p\rangle\langle p|\int dx'\,|x'\rangle\langle x'|\alpha(0)\rangle$$

$$= \int dx'\,dp\,\langle x|e^{-iP^2t/2m\hbar}|p\rangle\langle p|x'\rangle\langle x'|\alpha(0)\rangle$$

$$e^{-ip^2t/2m\hbar}$$

$$= \int dx'\,dp\; e^{-ip^2t/2m\hbar}\underbrace{\langle x|p\rangle\langle p|x'\rangle}_{\frac{e^{ip(x-x')/\hbar}}{2\pi\hbar}}\langle x'|\alpha(0)\rangle$$

$$= \int dx'\,dp\;\frac{e^{-ip^2t/2m\hbar + ip(x-x')/\hbar}}{2\pi\hbar}\;\langle x'|\alpha(0)\rangle$$

⑧

$$\int_{-\infty}^{\infty} dp \quad e^{-i\left(\frac{t}{2m\hbar}p^2 - \frac{x-x'}{\hbar}p\right)} = ?$$

$$p^2 \to p^2 - i\varepsilon, \qquad \varepsilon \sim 0$$

$$-i\left(\frac{t}{2m\hbar}(p^2 - i\varepsilon) - \frac{x-x'}{\hbar}p\right) = -i\left(\frac{t}{2m\hbar}p^2 - \frac{x-x'}{\hbar}p\right) - \varepsilon$$

ε will regulate the integral so we can compute the usual Fresnel-Gauss integral:

$$\int_{-\infty}^{\infty} dp \; e^{-\left(\frac{it}{2m\hbar}p^2 - \frac{i(x-x')}{\hbar}p\right)} = ?$$

$$A := \frac{it}{2m\hbar}, \qquad B := -\frac{i(x-x')}{\hbar}$$

$$Ap^2 + Bp = Ap^2 + 2\frac{B}{2\sqrt{A}}\sqrt{A}\,p + \frac{B^2}{4A} - \frac{B^2}{4A}$$

$$= \left(\sqrt{A}\,p + \frac{B}{2\sqrt{A}}\right)^2 - \frac{B^2}{4A}$$

$$= A\left(p + \frac{B}{2A}\right)^2 - \frac{B^2}{4A}$$

$$\int_{-\infty}^{\infty} dp \; e^{-(Ap^2+Bp)} = \int_{-\infty}^{\infty} dp \; e^{-A(p+B/2A)^2} \; e^{-B^2/4A}, \quad p \to p - \frac{B}{2A}$$

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$$= \int_{-\infty}^{\infty} dp\, e^{-Ap^2}\, e^{-B^2/4A}$$

$$= \sqrt{\frac{\pi}{A}}\, e^{-B^2/4A}$$

$$= \sqrt{\frac{\pi}{it/2m\hbar}}\, \exp \frac{\left(-\frac{i(x-x')}{\hbar}\right)^2}{4\frac{it}{2m\hbar}}$$

$$= \sqrt{\frac{2m\hbar\pi}{it}}\, \underbrace{\exp \frac{-\frac{(x-x')^2}{\hbar^2}}{\frac{2it}{m\hbar}}}_{e^{i(x-x')^2 m/2\hbar t}}$$

$$= \sqrt{\frac{2m\hbar\pi}{it}}\, e^{i(x-x')^2 m/2\hbar t}$$

$$\langle x|\alpha(t)\rangle = \int \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx' \sqrt{\frac{2m\hbar\pi}{it}}\, e^{i(x-x')^2 m/2\hbar t} \langle x'|\alpha(0)\rangle$$

$$= \sqrt{\frac{m}{2\pi\hbar i t}} \int_{-\infty}^{\infty} dx'\, e^{i(x-x')^2 m/2\hbar t} \sqrt{K}\, e^{-K|x'|}$$

(10)

$$= \sqrt{\frac{Km}{2\pi\hbar i t}} \left(\int_0^\infty dx' \, e^{i(x-x')^2 m/2\hbar t} \, e^{-Kx'} + \int_{-\infty}^0 dx' \, e^{i(x-x')^2 m/2\hbar t} \, e^{Kx'} \right) \to x' \to -x'$$

$$= \sqrt{\frac{Km}{2\pi\hbar i t}} \left(\int_0^\infty dx' \, e^{i(x-x')^2 m/2\hbar t} \, e^{-Kx'} + \int_0^\infty dx' \, e^{i(x+x')^2 m/2\hbar t} \, e^{-Kx'} \right)$$

$$= \sqrt{\frac{Km}{2\pi\hbar i t}} \left(\int_0^\infty dx' \, e^{-\frac{m}{2i\hbar t}(x-x')^2 - Kx'} + \int_0^\infty dx' \, e^{-\frac{m}{2i\hbar t}(x+x')^2 - Kx'} \right)$$

$$= \sqrt{\frac{Km}{2\pi\hbar i t}} \left(\int_0^\infty dx' \, e^{-\left(\frac{m}{2i\hbar t}(x-x')^2 + Kx'\right)} + \int_0^\infty dx' \, e^{-\left(\frac{m}{2i\hbar t}(x+x')^2 + Kx'\right)} \right)$$

$$\frac{m}{2i\hbar t}(x \pm x')^2 + Kx' = \frac{mx^2}{2i\hbar t} + \frac{m}{2i\hbar t}x'^2 \pm 2\frac{m}{2i\hbar t}xx' + Kx'$$

$$= \frac{m}{2i\hbar t}x'^2 + \left(K \pm \frac{mx}{i\hbar t}\right)x' + \frac{mx^2}{2i\hbar t}$$

$$=: C x'^2 + D_\pm x' + E$$

$$= Cx'^2 + 2\frac{D_\pm}{2\sqrt{C}}\sqrt{C}\,x' + \frac{D_\pm^2}{4C} - \frac{D_\pm^2}{4C} + E$$

$$= \left(\sqrt{C}x' + \frac{D_\pm}{2\sqrt{C}}\right)^2 + E - \frac{D_\pm^2}{4C}$$

$$= C\left(x' + \frac{D_\pm}{2C}\right)^2 + E - \frac{D_\pm^2}{4C}$$

$$\therefore \langle x|\alpha(t)\rangle = \sqrt{\frac{m}{2i\pi\hbar t}}\sqrt{\frac{Km}{2\pi\hbar i t}}$$

$$\times\left(\int_0^\infty dx'\, e^{-C(x'+D_-/2C)^2}\, e^{-E+D_-^2/4C} \quad \Rightarrow x' \to x' - \frac{D_-}{2C}\right.$$

$$\left. + \int_0^\infty dx'\, e^{-C(x'+D_+/2C)^2}\, e^{-E+D_+^2/4C}\right) \Rightarrow x' \to x' - \frac{D_+}{2C}$$

$$= \sqrt{\frac{Km^2}{(2\pi i\hbar t)^2}} \left(\int_{-D_-/2C}^{\infty} dx'\, e^{-Cx'^2}\, e^{-E+D_-^2/4C} \right.$$

$$\left. + \int_{-D_+/2C}^{\infty} dx'\, e^{-Cx'^2}\, e^{-E+D_+^2/4C} \right)$$

$$= \sqrt{\frac{Km^2}{(2\pi i\hbar t)^2}} \left[\left(\int_0^{\infty} dx'\, e^{-Cx'^2} + \overbrace{\int_{-D_-/2C}^{0} dx'\, e^{-Cx'^2}}^{x' \to -x'} \right) e^{-E+D_-^2/4C} \right.$$

$$\left. + \left(\int_0^{\infty} dx'\, e^{-Cx'^2} + \overbrace{\int_{-D_+/2C}^{0} dx'\, e^{-Cx'^2}}^{x' \to -x'} \right) e^{-E+D_+^2/4C} \right]$$

$$= \frac{\sqrt{K}\, m}{2\pi i\hbar t} \left(e^{-E+D_-^2/4C}\, \frac{1}{2}\sqrt{\frac{\pi}{C}} + e^{-E+D_-^2/4C} \int_0^{D_-/2C} dx'\, e^{-Cx'^2} \right.$$

$$\left. + e^{-E+D_+^2/4C}\, \frac{1}{2}\sqrt{\frac{\pi}{C}} + e^{-E+D_+^2/4C} \int_0^{D_+/2C} dx'\, e^{-Cx'^2} \right)$$

$$= \frac{\sqrt{K}\, m}{2\pi i\hbar t} \left(\frac{1}{2}\sqrt{\frac{\pi}{C}} \left(e^{-E+D_-^2/4C} + e^{-E+D_+^2/4C} \right) \right.$$

$$+ e^{-E+D_-^2/4C}\, \frac{1}{2}\sqrt{\frac{\pi}{C}}\, \mathrm{erf}\left(\frac{D_-}{2\sqrt{C}} \right)$$

$$\left. + e^{-E+D_+^2/4C}\, \frac{1}{2}\sqrt{\frac{\pi}{C}}\, \mathrm{erf}\left(\frac{D_+}{2\sqrt{C}} \right) \right)$$

(13)

$$\therefore \boxed{\langle x|\alpha(t)\rangle = \frac{\sqrt{K}\, m}{4\pi i\hbar t}\sqrt{\frac{\pi}{C}} \times \left(e^{-E + D_+^2/4C}\left(1 + \operatorname{erf}\left(\frac{D_+}{2\sqrt{C}}\right)\right) + e^{-E + D_-^2/4C}\left(1 + \operatorname{erf}\left(\frac{D_-}{2\sqrt{C}}\right)\right)\right)}$$

where

$$K = \frac{m\lambda}{\hbar^2}$$

$$C = \frac{m}{2i\hbar t}$$

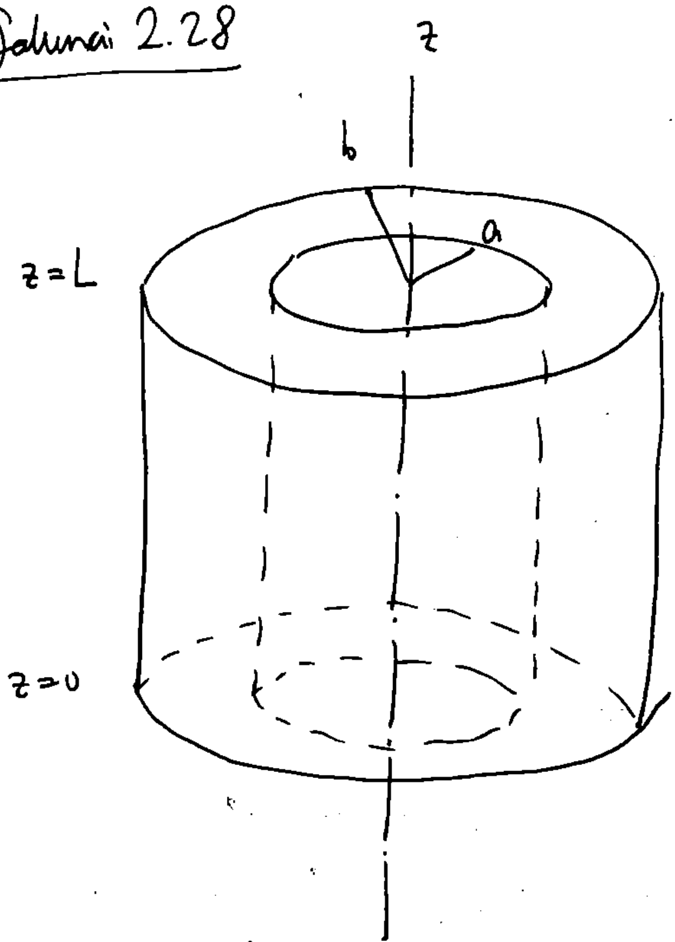
$$D_\pm = K \pm \frac{mx}{i\hbar t}$$

$$E = \frac{mx^2}{2i\hbar t}$$

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x du\, e^{-u^2}$$

(14)

Soluzioni 2.28



particle is free in the region

$a < s < b$

$0 < z < L$

$0 < \varphi < 2\pi$

(15)

(a) $H\phi(\vec{x}) = E\phi(\vec{x})$

$$H = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

For a geometry whose line element is given by

$$dl^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

the laplacian is defined as follows.

$$\nabla^2 := \frac{1}{\sqrt{g}}\partial_i\left((g^{-1})_{ij}\sqrt{g}\,\partial_j\right)$$

where

$$g_{ij} \doteq \begin{pmatrix} h_1^2 & & \\ & h_2^2 & \\ & & h_3^2 \end{pmatrix}$$

$$g_{ij}^{-1} = \begin{pmatrix} 1/h_1^2 & & \\ & 1/h_2^2 & \\ & & 1/h_3^2 \end{pmatrix}$$

$$g = \det g_{ij} = h_1^2 h_2^2 h_3^2$$

$$\therefore \nabla^2 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_1h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_2h_3}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2h_3}{h_3}\frac{\partial}{\partial u_3}\right)\right)$$

In cylindrical coordinates, we have

$$dl^2 = ds^2 + s^2 d\varphi^2 + dz^2$$

$$\therefore g_{ij} = \begin{pmatrix} 1 & & \\ & s^2 & \\ & & 1 \end{pmatrix}, \quad (g^{-1})_{ij} = \begin{pmatrix} 1 & & \\ & 1/s^2 & \\ & & 1 \end{pmatrix}, \quad \det g_{ij} = s^2$$

$$\therefore \nabla^2 = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{s}\frac{\partial}{\partial \varphi}\left(\frac{1}{s^2}s\frac{\partial}{\partial \varphi}\right) + \frac{1}{s}\frac{\partial}{\partial z}\left(s\frac{\partial}{\partial z}\right)$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(17)

$$-\frac{\hbar^2}{2m}\nabla^2\phi = E\phi$$

$$\nabla^2\phi = -\frac{2mE}{\hbar^2}\phi =: -k^2\phi$$

$$\phi(\vec{x}) = S(s)\,\Phi(\varphi)\,Z(z)$$

$$\nabla^2\phi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\phi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$= \Phi Z\,\frac{(sS')'}{s} + SZ\,\frac{1}{s^2}\,\Phi'' + S\Phi\,Z''$$

$$\Phi Z\,\frac{(sS')'}{s} + SZ\,\frac{1}{s^2}\Phi'' + S\Phi\,\frac{Z''}{z^2} = -k^2 S\Phi Z \quad \Big\backslash\ \frac{1}{S\Phi Z}$$

$$\frac{(sS')'}{sS} + \frac{1}{s^2}\underbrace{\frac{\Phi''}{\Phi}}_{-m^2} + \underbrace{\frac{Z''}{Z}}_{-n^2} = -k^2$$

$$\Phi(\varphi) = e^{im\varphi}\ ; \quad \Phi(\varphi+2\pi) = \Phi(\varphi) \Rightarrow m \in \mathbb{Z}$$

$$Z(z) = A\sin nz + B\cos nz$$

$$Z(0) = 0 \Rightarrow B = 0$$

$$Z(L) = 0 \Rightarrow n = \frac{l\pi}{L}, \quad l \in \mathbb{Z}^+$$

$$\therefore \; Z(z) = A \sin \frac{l \pi z}{L}$$

$$\frac{(sS')'}{sS} - \frac{m^2}{s^2} - n^2 = -k^2 \quad \Big| \; s^2 S$$

$$\underbrace{s(sS')'}_{s^2 S'' + sS'} - m^2 S - n^2 s^2 S = -k^2 s^2 S$$

$$s^2 S'' + sS' + \left((k^2 - n^2) s^2 - m^2 \right) S = 0 \quad \text{Bessel eqn}$$

$$S(s) = A J_m\left(\sqrt{k^2-n^2}\, s \right) + B N_m\left(\sqrt{k^2-n^2}\, s \right)$$

Both Bessel I and Bessel II will be used $\because$ the region is away from both extremes (0 and ∞).

$$\begin{cases} S(a) = A J_m\left(\sqrt{k^2-n^2}\, a \right) + B N_m\left(\sqrt{k^2-n^2}\, a \right) = 0 & (*) \\ S(b) = A J_m\left(\sqrt{k^2-n^2}\, b \right) + B N_m\left(\sqrt{k^2-n^2}\, b \right) = 0 & (**) \end{cases}$$

$$(*) \Rightarrow A = - \frac{B N_m\left(\sqrt{k^2-n^2}\, a \right)}{J_m\left(\sqrt{k^2-n^2}\, a \right)}$$

$$(**) \Rightarrow -\frac{B N_m\left(\sqrt{k^2-n^2}\, a \right)}{J_m\left(\sqrt{k^2-n^2}\, a \right)} \Bigg|_{J_m\left(\sqrt{k^2-n^2}\, b\right)} + B N_m\left(\sqrt{k^2-n^2}\, b \right) = 0$$

(19)

$$J_m(\sqrt{k^2-n^2}\,a)\,N_m(\sqrt{k^2-n^2}\,b) - J_m(\sqrt{k^2-n^2}\,b)\,N_m(\sqrt{k^2-n^2}\,a) = 0$$

Assume $\exists\ \beta_{ml}$: β_{ml} solve the eqn above. Then

$$\beta_{mj} = \sqrt{k^2 - n^2}\,, \quad j \in \mathbb{Z}^+ \quad (\exists\ \infty \text{ many roots of that eqn})$$

$$\therefore k^2 = \beta_{mj}^2 + n^2 = \beta_{mj}^2 + \left(\frac{l\pi}{L}\right)^2$$

$$\frac{2mE}{\hbar^2} = \beta_{mj}^2 + \left(\frac{l\pi}{L}\right)^2$$

$$\boxed{E_{lmj} = \frac{\hbar^2}{2m}\left(\beta_{mj}^2 + \frac{l^2\pi^2}{L^2}\right)}$$

(20)

Sakurai 2.30

Continuity eqn for ψ :

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = \frac{p^2}{2m}\psi + V\psi$$

$$= -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi, \quad \text{assume } V \text{ is real valued}$$

$$\psi^*(\text{eqn}) - (\text{eqn})^*\psi = 0$$

$$\psi^*\left(i\hbar\dot{\psi} + \frac{\hbar^2}{2m}\psi'' - \cancel{V\psi}\right) - \left(-i\hbar\dot{\psi}^* + \frac{\hbar^2}{2m}\psi^{*''} - \cancel{V\psi^*}\right)\psi = 0$$

$$i\hbar\underbrace{\left(\psi^*\dot{\psi} + \dot{\psi}^*\psi\right)}_{\frac{\partial}{\partial t}\psi^*\psi} + \frac{\hbar^2}{2m}\underbrace{\left(\psi^*\psi'' - \psi^{*''}\psi\right)}_{} = 0$$

$$= (\psi^*\psi')' - \cancel{\psi^{*'}\psi'} - (\psi^{*'}\psi)' + \cancel{\psi^{*'}\psi'}$$

$$= (\psi^*\psi' - \psi^{*'}\psi)'$$

$$i\hbar\frac{\partial|\psi|^2}{\partial t} + \frac{\hbar^2}{2m}\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) = 0$$

$$\frac{\partial|\psi|^2}{\partial t} + \frac{\hbar}{2im}\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) = 0$$

(21)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\therefore \vec{J} = \frac{\hbar}{2im} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right)$$

$$= \frac{\hbar}{2im} 2i \, \mathrm{Im} \, \psi^* \vec{\nabla} \psi$$

$$= \frac{\hbar}{m} \mathrm{Im} \, \psi^* \vec{\nabla} \psi \quad =$$

$$z = a + ib \implies b = \mathrm{Im}\, z = \mathrm{Re} \frac{z}{i}$$

$$= \frac{\hbar}{m} \mathrm{Re} \frac{\psi^* \vec{\nabla} \psi}{i}$$

$$= \mathrm{Re} \, \psi^* \frac{\frac{\hbar \vec{\nabla}}{i}}{m} \psi$$

$$= \mathrm{Re} \, \psi^* \frac{\vec{P}}{m} \psi$$

$$\boxed{\vec{J} = \mathrm{Re} \, \psi^* \vec{V} \psi}$$ makes sense $\because$ this is nonrelat phys.

H-atom:

$$\psi(\vec{x}) \sim \underbrace{\text{Laguerre polynomials}}_{\text{real}} \times \underbrace{\text{spherical harmonics}}_{\text{complex}}$$

(22)

$$\psi^* \vec{\nabla}_r \psi = \underbrace{\psi^*}_{RC^*} \frac{\hbar}{im} \underbrace{}_{\frac{\partial}{\partial r}} \underbrace{\psi}_{RC} \Rightarrow$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

R: real part, depending only on r

C: complex part, depending only on angles

$$\Rightarrow \underbrace{RC^* \frac{\hbar}{im} C}_{|C|^2} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) R$$

$$= \frac{1}{i}\,(\text{real})$$

$\therefore$ $J_r = 0$ $\quad\therefore$ no radial flow

$$\psi^* \nabla_\theta \psi = \psi^* \frac{\hbar}{im} \frac{1}{r} \frac{\partial}{\partial \theta} \psi$$

$$= R C^* \frac{\hbar}{im} \frac{1}{r} \frac{\partial}{\partial \theta} R C$$

Angular part contains complex unit only via $e^{im\varphi}$ so, this bit also gives 0.

23

$$\psi^* V_\varphi \psi = \psi^* \frac{\hbar}{i\mu} \frac{1}{r\sin\theta} \frac{\partial}{\partial\varphi} \psi$$

real $\times e^{im\varphi}$

$$= R e^{-im\varphi} \frac{\hbar}{i\mu} \frac{1}{r\sin\theta} \frac{\partial}{\partial\varphi} R e^{im\varphi}$$

$$= \underbrace{(R^* e^{-im\varphi})(R e^{im\varphi})}_{|\psi|^2} \, im \, \frac{\hbar}{i\mu r\sin\theta}$$

$$= \frac{\hbar m |\psi|^2}{\mu r\sin\theta}$$

$$\therefore \vec{J} = J_\varphi \hat{\varphi}$$

$$\boxed{\vec{J} = \frac{\hbar m |\psi|^2}{\mu r\sin\theta} \hat{\varphi}}$$

Quote of the week:

"If you haven't found something strange during the day, it hasn't been much of a day."

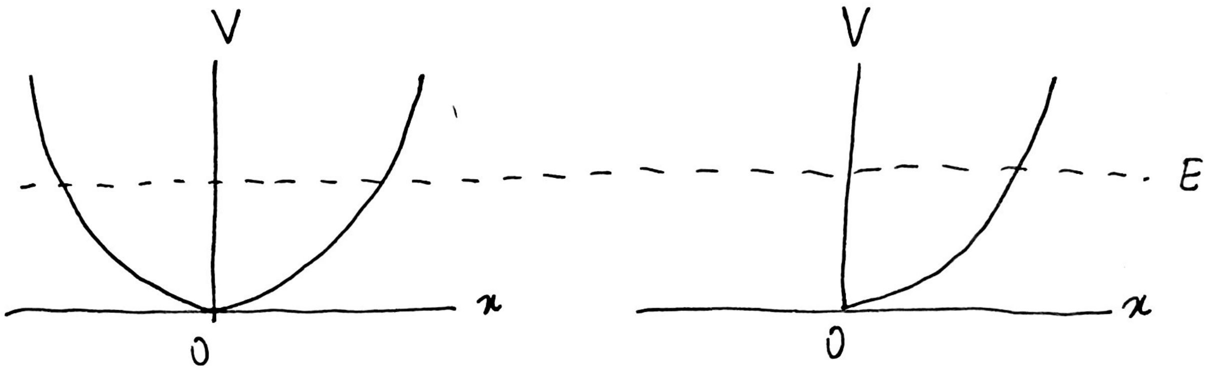
- Wheeler

①

507 RECIT 4

Sakurai 2.22

$$V(x) = \begin{cases} \frac{1}{2}kx^2, & x>0 \\ \infty, & x<0 \end{cases} \qquad \text{"half oscillator"}$$

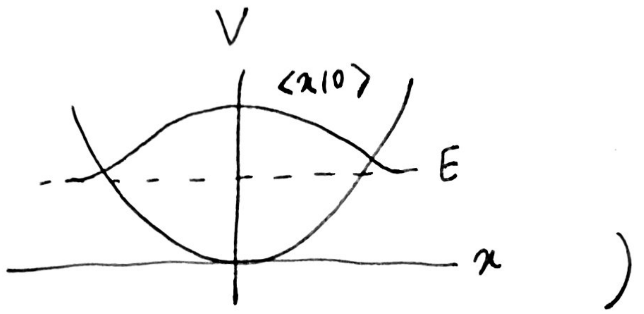


Since we will be solving the same Schr w/ the same pot in the region $x>0$, we should have the same solutions. The major difference will be in the boundary conditions. For "half oscillator", we have to impose

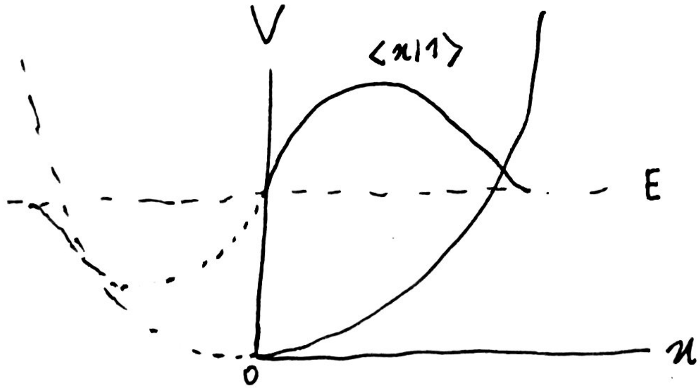
$$\langle x|n\rangle\Big|_{x=0} = \langle 0|n\rangle = 0 \quad \forall n$$

Since the full oscillator has even (symmetric) pot, the eigenfunctions of the Hamiltonian should be either even or odd about the origin ($\because$ parity is conserved). Since the ground state of the full oscillator is even, (see

(2)



by induction all the solutions w/ n even should behave like this. $\therefore$ The half oscillator has the odd solutions of the full oscillator to satisfy the BCs:



Therefore the ground state of the half oscillator is $|1\rangle$. Since our domain has also change, we need to redenive $\langle x|0\rangle$ from scratch:

$$a_\pm := \sqrt{\frac{1}{2m\omega\hbar}}\,(m\omega X \mp iP)$$

$$a_-|0\rangle = 0$$

$$\langle x|a_-|0\rangle = \langle x|\,\frac{m\omega X \mp iP}{\sqrt{2m\omega\hbar}}\,|0\rangle$$

$$= \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\langle x|X|0\rangle + i\langle x|P|0\rangle\right)$$

③

$$= \frac{1}{\sqrt{2m\omega\hbar}} \left(m\omega x \langle x|0\rangle + i \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|0\rangle \right)$$

$$= 0$$

$$m\omega x \langle x|0\rangle + \hbar \frac{\partial}{\partial x} \langle x|0\rangle = 0$$

$$\frac{\partial \langle x|0\rangle}{\langle x|0\rangle} = -\frac{m\omega}{\hbar} x \, \partial x =: -\frac{x \, \partial x}{x_0^2}, \quad x_0^2 := \frac{\hbar}{m\omega}$$

$$\therefore \langle x|0\rangle = N e^{-x^2/2x_0^2}$$

- $a_+ |0\rangle = |1\rangle$

$$\therefore \langle x|1\rangle = \langle x|a_+|0\rangle$$

$$= \langle x| \frac{m\omega X - iP}{\sqrt{2m\omega\hbar}} |0\rangle$$

$$= \frac{1}{\sqrt{2m\omega\hbar}} \left(m\omega \langle x|X|0\rangle - i \langle x|P|0\rangle \right)$$

$$= \frac{1}{\sqrt{2m\omega\hbar}} \left(m\omega x \langle x|0\rangle - i \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|0\rangle \right)$$

$$= \frac{1}{\sqrt{2m\omega\hbar}} \left(m\omega x \langle x|0\rangle - \hbar \frac{\partial}{\partial x} \langle x|0\rangle \right)$$

$$= \frac{1}{\sqrt{2m\omega\hbar}} \hbar \left(\frac{m\omega}{\hbar} x \langle x|0\rangle - \frac{\partial}{\partial x} \langle x|0\rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{m\omega}{\hbar} x \langle x|0\rangle - \frac{\partial}{\partial x} \langle x|0\rangle \right)$$

$$= \sqrt{\frac{x_0^2}{2}} \left(\frac{x}{x_0^2} \langle x|0\rangle - \frac{\partial}{\partial x} \langle x|0\rangle \right)$$

$$\frac{\partial}{\partial x} \langle x|0\rangle = \frac{\partial}{\partial x} N e^{-x^2/2x_0^2}$$

$$= -N \frac{x}{x_0^2} e^{-x^2/2x_0^2} = -\frac{x}{x_0^2} \langle x|0\rangle$$

$$\therefore \langle x|1\rangle = \sqrt{\frac{x_0^2}{2}} \left(\frac{x}{x_0^2} \langle x|0\rangle + \frac{x}{x_0^2} \langle x|0\rangle \right)$$

$$= \sqrt{\frac{x_0^2}{2}} \; \frac{2x}{x_0^2} \langle x|0\rangle$$

$$= \frac{\sqrt{2}\, x}{x_0} N e^{-x^2/2x_0^2}$$

$$\langle 1|1\rangle = 1$$

$$= \int_0^\infty dx \, \langle 1|x\rangle \langle x|1\rangle$$

$$= \int_0^\infty dx \, \frac{2N^2}{x_0^2} \, x^2 e^{-x^2/x_0^2}$$

$$= \frac{2N^2}{x_0^2} \int_0^\infty dx \, x^2 e^{-x^2/x_0^2}$$

(5)

$$\int_0^\infty dr\, r^2 e^{-\alpha r^2} = ?$$

$$\int d^3r\, e^{-\alpha \vec{r}^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, e^{-\alpha(x^2+y^2+z^2)}$$

$$= \left(\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \right)^3 = \left(\sqrt{\frac{\pi}{\alpha}} \right)^3$$

$$\int d^3r\, e^{-\alpha \vec{r}^2} = 4\pi \int_0^\infty dr\, r^2 e^{-\alpha r^2}$$

$$\therefore \int_0^\infty dr\, r^2 e^{-\alpha r^2} = \frac{1}{4\pi} \left(\sqrt{\frac{\pi}{\alpha}} \right)^3$$

$$\therefore \int_0^\infty dx\, x^2\, e^{-x^2/x_0^2} = \frac{1}{4\pi} \left(x_0^2 \pi \right)^{3/2}$$

$$\therefore \langle 1|1 \rangle = \frac{2N^2}{x_0^2} \frac{1}{4\pi} \left(x_0^2 \pi \right)^{3/2}$$

$$\therefore N^2 = \frac{2 \cancel{4} \pi x_0^2}{\cancel{4} \left(x_0^2 \pi \right)^{3/2}}$$

Ground state of half oscillator

$$\therefore \boxed{\langle x|1 \rangle = \sqrt{\frac{2\pi x_0^2}{\left(x_0^2 \pi \right)^{3/2}}} \frac{\sqrt{2}\, x}{x_0} e^{-x^2/2x_0^2}} \qquad 0 < x < \infty$$

$$\boxed{E_1 = \hbar\omega \left(n + \frac{1}{2} \right) \Big|_{n=1} = \frac{3}{2} \hbar\omega}$$ energy of ground state

(6)

$$\langle X^2 \rangle = \langle 1 | X^2 | 1 \rangle$$

$$= \int_0^\infty dx \, \langle 1 | x \rangle \, x^2 \, \langle x | 1 \rangle$$

$$= \frac{2\pi x_0^2}{(x_0^2 \pi)^{3/2}} \, \frac{2}{x_0^2} \int_0^\infty dx \, x^4 e^{-x^2/x_0^2}$$

$$\int_0^\infty dx \, x^2 e^{-\alpha x^2} = \frac{1}{4\pi} \left(\sqrt{\frac{\pi}{\alpha}} \right)^3 = \frac{\pi^{3/2}}{4\pi} \alpha^{-3/2} \quad \Big| -\frac{\partial}{\partial \alpha}$$

$$\int_0^\infty dx \, x^4 e^{-\alpha x^2} = \frac{\pi^{3/2}}{4\pi} \, \frac{3}{2} \, \alpha^{-5/2} = \frac{3\pi^{3/2}}{8\pi} \alpha^{-5/2}$$

$$\therefore \int_0^\infty dx \, x^4 e^{-x^2/x_0^2} = \frac{3\pi^{3/2}}{8\pi} (x_0^2)^{5/2} = \frac{3\pi^{3/2} x_0^5}{8\pi}$$

$$\therefore \langle X^2 \rangle = \frac{2\pi x_0^2}{(x_0^2 \pi)^{3/2}} \, \frac{2}{x_0^2} \, \frac{3\pi^{3/2} x_0^5}{8\pi}$$

$$\boxed{\langle X^2 \rangle = \frac{3 x_0^2}{2}}$$

(7)

Sakurai 2.27

"Density of states" is defined as the Jacobian of the transformation from phase space to the "energy space."

$$\frac{d^3x\, d^3p}{h^3} = D(E)\, dE$$

For a free particle, $\nexists$ any dependence on x, so d^3x can be directly integrated to give V, volume.

$$\frac{V}{h^3} d^3p = D(E)\, dE$$

For a free particle, $E = \vec{p}^{\,2}/2m$, so there is no angular dependence, either:

$$d^3p = |\vec{p}|^2 d|\vec{p}|\, d\Omega = 4\pi |\vec{p}|^2 d|\vec{p}|$$

$$\therefore \frac{V}{h^3} 4\pi |\vec{p}|^2 d|\vec{p}| = D(E)\, dE$$

$$\therefore D(E) = \frac{V}{h^3} 4\pi \underbrace{|\vec{p}|^2}_{2mE} \underbrace{\left|\frac{d|\vec{p}|}{dE}\right|}_{\left|\frac{d}{dE}(2mE)^{1/2}\right|}$$

$$= \frac{V}{h^3} 4\pi\, 2mE\, \frac{1}{\cancel{2}} (2mE)^{-1/2}\, \cancel{2}m$$

$$= \frac{V}{h^3} \; 8\pi m^2 E \, (2mE)^{-1/2}$$

$$\boxed{D(E) = \frac{V}{h^3} \frac{8\pi m^2}{\sqrt{2m}} \sqrt{E}} \qquad 3D$$

From this, you can switch to density of states in terms of $\vec{k}$ or $\vec{p}$ or whatever parameter you want to control: (As long as you know its dependence on E)

$$D(E)\,dE = D(\lambda)\,d\lambda$$

(λ: not wavelength but arbitrary parameter)

$$D(\lambda) = D(E(\lambda)) \left| \frac{dE}{d\lambda} \right|$$

Why the absolute value? $\because$ the Jacobian is an intrinsically positive quantity.

In 2D:

$$\frac{d^2x \, d^2p}{h^2} = D(E)\,dE$$

$$\frac{A}{h^2} \underbrace{|\vec{p}|}_{\sqrt{2mE}} d|\vec{p}| \, d\varphi = D(E)\,dE$$

$$D(E) = \frac{A}{h^2} \sqrt{2mE} \underbrace{\left| \frac{d|\vec{p}|}{dE} \right|}_{\frac{m}{\sqrt{2mE}}} d\varphi = \frac{A}{h^2} \cancel{\sqrt{2mE}} \frac{m}{\cancel{\sqrt{2mE}}} d\varphi$$

indep of E

(9)

in 1D:

$$\frac{dx\,dp}{h} = D(E)\,dE$$

$$\frac{L}{h}\,dp = D(E)\,dE$$

$$D(E) = \frac{L}{h}\left|\frac{dp}{dE}\right| = \frac{L}{h}\,\frac{m}{\sqrt{2mE}}$$

$$D_{3D}(E) \propto E^{1/2}$$

$$D_{2D}(E) \propto E^{0}$$

$$D_{1D}(E \propto E^{-1/2}$$

(1)

Sakurai 2.32

$$K(\vec{x}', t'; \vec{x}, t) = \langle \vec{x}' | e^{-\frac{i}{\hbar} H (t'-t)} | \vec{x} \rangle$$

$$Z := \int d^3x \, \langle \vec{x} | e^{-\frac{i}{\hbar} Ht} | \vec{x} \rangle \Big|_{\frac{it}{\hbar} \to \beta}$$

$$= \int d^3x \, \langle \vec{x} | e^{-\beta H} | \vec{x} \rangle$$

$$= \int d^3x \, \langle \vec{x} | e^{-\beta H} \sum_n | n \rangle \langle n | \vec{x} \rangle$$

$$= \sum_n e^{-\beta E_n} \underbrace{\int d^3x \, \langle \vec{x} | n \rangle \langle n | \vec{x} \rangle}_{\substack{1 \text{ if the eigenkets are} \\ \text{normalized.} \\ \text{(or at least some const, } N^2)}}$$

$$= \sum_n e^{-\beta E_n}$$

As $\beta \to \infty$, less and less terms contribute to the sum, so we get

$$Z = e^{-\beta E_0}$$

where E_0 is the ground-state energy.

(11) One way to isolate (or extract) E_0 is to take derivatives:

$$\frac{\partial Z}{\partial \beta} = -E_0 e^{-\beta E_0} = -Z E_0$$

$$\therefore \boxed{E_0 = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}}$$ in the limit $\beta \to \infty$

Particle in a box: For a box of size $[0, L]$,

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}, \qquad n \in \mathbb{Z}^+$$

$$=: \varepsilon n^2$$

$$Z = \sum_{n \geq 1} e^{-\beta E_n} = \sum_{n \geq 1} e^{-n^2 \beta \varepsilon}$$

For $\beta \to \infty$,

$$Z = e^{-\beta\varepsilon} + e^{-4\beta\varepsilon} + \dots = e^{-\beta\varepsilon} + \mathcal{O}(e^{-\beta\varepsilon})^4 \simeq e^{-\beta\varepsilon}$$

$$\frac{\partial Z}{\partial \beta} = -\varepsilon e^{-\beta\varepsilon} \Rightarrow E_1 = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \varepsilon$$ ✓

Quote of the week:

"I think I can safely say that nobody understands quantum mechanics."

– Feynman

1

EXTRA 1

$$\langle p|\underset{\sim}{X}|\alpha\rangle = \int dx\, \langle p|\underset{\sim}{X}|x\rangle\langle x|\alpha\rangle$$

$$= \int dx\, dp'\, \langle p|x\rangle\, x\, \langle x|p'\rangle\langle p'|\alpha\rangle$$

$$= \int dx\, dp'\, \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\, \frac{e^{ip'x/\hbar}}{\sqrt{2\pi\hbar}}\, x\,\langle p'|\alpha\rangle$$

$$= \int dp' \left(\int \frac{dx}{2\pi\hbar}\, x e^{i(p'-p)x/\hbar} \right) \langle p'|\alpha\rangle$$

$$= \int dp' \left(-\frac{\hbar}{i}\frac{\partial}{\partial p} \int \frac{dx}{2\pi\hbar}\, e^{i(p'-p)x/\hbar} \right) \langle p'|\alpha\rangle$$

$$= -\frac{\hbar}{i}\frac{\partial}{\partial p} \int dp'\, \delta(p'-p)\, \langle p'|\alpha\rangle$$

$$\boxed{\langle p|\underset{\sim}{X}|\alpha\rangle = -\frac{\hbar}{i}\frac{\partial}{\partial p}\langle p|\alpha\rangle}$$

$$i\hbar \frac{\partial}{\partial t}|\alpha(t)\rangle = H|\alpha(t)\rangle$$

$$H = \frac{P^2}{2m} + V(\underset{\sim}{X}), \qquad V(\underset{\sim}{X}) = -qE\underset{\sim}{X}$$

$$\therefore \frac{\partial H}{\partial t} = 0$$

$$\therefore H|\alpha\rangle = E|\alpha\rangle$$

$$\langle p|H|\alpha\rangle = E\langle p|\alpha\rangle$$

(2)

$$\langle p | \frac{p^2}{2m} - qE x | \alpha \rangle = E \langle p | \alpha \rangle$$

$$\frac{p^2}{2m} \langle p|\alpha\rangle - qE \frac{-\hbar}{i} \frac{\partial}{\partial p} \langle p|\alpha\rangle = \varepsilon \langle p|\alpha\rangle$$

$$\langle p|\alpha\rangle' \frac{qE\hbar}{i} + \langle p|\alpha\rangle \frac{p^2}{2m} = \varepsilon \langle p|\alpha\rangle \quad \Big| \; \frac{i}{qE\hbar}$$

$$\langle p|\alpha\rangle' + \frac{ip^2}{2mqE\hbar} \langle p|\alpha\rangle = \frac{i\varepsilon}{qE\hbar} \langle p|\alpha\rangle$$

Canonical form:

$$y' + (A x^2 + B) y = 0$$

$$\frac{dy}{y} = -(Ax^2 + B)\, dx$$

$$\ln y = -\left(\frac{Ax^3}{3} + Bx\right) + \ln y(0)$$

$$y(x) = y(0)\, e^{-(Ax^3/3 + Bx)}$$

Put
$$x \to p$$
$$y \to \langle p|\alpha\rangle$$
$$A \to \frac{i}{2mqE\hbar}$$
$$B \to \frac{i\varepsilon}{qE\hbar}.$$

Normalization (or initial condition) is open to discussion.

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EXTRA 2

2.16

(See Recit. 3 notes)

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(a) $X = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$

$|\alpha\rangle = a|0\rangle + b|1\rangle, \quad |a|^2 + |b|^2 = 1$

$$\langle X\rangle = (a^*\langle 0| + b^*\langle 1|)\sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)(a|0\rangle + b|1\rangle)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,(a^*\langle 0| + b^*\langle 1|)(a|1\rangle + \sqrt{2}\,b|2\rangle + b|0\rangle)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,(a^*b + b^*a)$$

Maximize $a^*b + b^*a$ subject to the constraint $|a|^2 + |b|^2 - 1 = 0$:

$f := a^*b + b^*a + \lambda(a^*a + b^*b - 1)$ → the trick is, treat a and a^* (and b and b^*) as indep. parameters.

$$\frac{\partial f}{\partial a} = b^* + \lambda a^* = 0$$

$$\frac{\partial f}{\partial a^*} = b + \lambda a = 0$$

$$\frac{\partial f}{\partial b} = a^* + \lambda b^* = 0$$

$$\frac{\partial f}{\partial b^*} = a + \lambda b = 0$$

④

$$\left.\begin{aligned} a^* &= -\frac{1}{d} b^* \\ a^* &= -d b^* \end{aligned}\right\} \quad -\frac{1}{d} = -d \quad \therefore \quad d = \pm 1$$

$$\left.\begin{aligned} a &= -\frac{1}{d} b \\ a &= -db \end{aligned}\right\} \text{ same}$$

$$a^* a + b^* b = 1$$

$$(-db^*)(-db) + b^* b = 1$$

$$\underbrace{d^2}_{1} b^* b + b^* b = 1$$

$$2 b^* b = 1$$

$$\therefore |b| = \frac{1}{\sqrt{2}}$$

$$\therefore |a| = \frac{1}{\sqrt{2}}$$

Assume $a, b \in \mathbb{R}^+$:

$$\boxed{|\alpha\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}}$$

("Construct $\underline{\underline{a}}$ linear combination...")

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(b) $$|\alpha(0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|\alpha(t)\rangle = e^{-iHt/\hbar} |\alpha(0)\rangle$$

$$\boxed{|\alpha(t)\rangle = \frac{e^{-iE_0t/\hbar}|0\rangle + e^{-iE_1t/\hbar}|1\rangle}{\sqrt{2}}, \quad t > 0, \quad E_n = \hbar\omega\left(n+\frac{1}{2}\right)}$$

In Schr. pic:

$$\langle X\rangle = \langle\alpha(t)|X|\alpha(t)\rangle$$

$$= \frac{e^{iE_0t/\hbar}\langle 0| + e^{iE_1t/\hbar}\langle 1|}{\sqrt{2}} \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$$

$$\times \frac{e^{-iE_0t/\hbar}|0\rangle + e^{-iE_1t/\hbar}|1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{iE_0t/\hbar}\langle 0| + e^{iE_1t/\hbar}\langle 1|\right)$$

$$\times\left(e^{-iE_0t/\hbar}|1\rangle + e^{-iE_1t/\hbar}|0\rangle + (\quad)|2\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i(E_0-E_1)t/\hbar} + e^{-i(E_0-E_1)t/\hbar}\right)$$

$$\boxed{\langle X\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\frac{(E_0-E_1)t}{\hbar}}$$

⑥

In Heisenberg pic:

$$|\alpha(t)\rangle = |\alpha(0)\rangle$$

$$X(t) = \mathcal{U}^\dagger(t) X \mathcal{U}(t)$$

$$= e^{iHt/\hbar} X e^{-iHt/\hbar}$$

→ compute it either from Heis. eqn of motion or from Baker-Hausdorff formula

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2$$

$$[X, H] = \frac{1}{2m}[X, P^2] = \frac{i\hbar P}{m}$$

$$\therefore \dot{X} = \frac{1}{m} P$$

$$[P, H] = \frac{1}{2} m\omega^2 [P, X^2] = \frac{m\omega^2}{2}(-2i\hbar X)$$

$$= -i\hbar m\omega^2 X$$

$$\therefore \dot{P} = - m\omega^2 X$$

$$\therefore \ddot{X} = -\omega^2 X \quad \text{et} \quad \ddot{P} = -\omega^2 P$$

$$X(t) = A\cos\omega t + B\sin\omega t$$

$$P(t) = C\cos\omega t + D\sin\omega t$$

$$\dot{X} = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$\dot{P} = -\omega C\sin\omega t + \omega D\cos\omega t$$

$$-\omega A \sin\omega t + \omega B \cos\omega t = \frac{1}{m} C \cos\omega t + \frac{1}{m} D \sin\omega t$$

Since sin and cos are linearly indep.,

$$-\omega A = \frac{1}{m} D$$

$$\omega B = \frac{1}{m} C$$

Meantime,

$$A = X(0), \qquad C = P(0)$$

$$\therefore D = -m\omega X(0) \quad \text{et} \quad B = \frac{1}{m\omega} P(0)$$

$$\therefore X(t) = X \cos\omega t + \frac{1}{m\omega} P \sin\omega t$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a) \cos\omega t + \frac{1}{m\omega} i \sqrt{\frac{\hbar m\omega}{2}} (a^\dagger - a) \sin\omega t$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \Big(a_+ (\cos\omega t + i\sin\omega t) + a_- (\cos\omega t - i \sin\omega t) \Big)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(e^{i\omega t} a_+ + e^{-i\omega t} a_- \right)$$

$$\therefore \langle X \rangle = \langle \alpha(0) | X(t) | \alpha(0) \rangle$$

$$= \frac{\langle 0| + \langle 1|}{\sqrt{2}} \sqrt{\frac{\hbar}{2m\omega}} \left(e^{i\omega t} a_+ + e^{-i\omega t} a_- \right) \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \Big(\langle 0| + \langle 1| \Big) \Big(e^{i\omega t} |1\rangle + (\ )|2\rangle + e^{-i\omega t} |0\rangle \Big)$$

$$= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \underbrace{\left(e^{-i\omega t} + e^{i\omega t} \right)}_{2\cos\omega t}$$

⑧

$$= \cancel{\frac{1}{2}} \sqrt{\frac{\hbar}{2m\omega}} \cos\omega t \qquad \text{Same result.}$$

(c) Assume Schr. pic:

$$X^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2, \quad [a_\pm, a_\mp] = \mp 1$$

$$= \frac{\hbar}{2m\omega}\Big(a_+^2 + a_-^2 + \underbrace{a_+a_-}_{N} + \underbrace{a_-a_+}_{\underbrace{a_+a_-}_{N} + \underbrace{[a_-, a_+]}_{1}}\Big)$$

$$= \frac{\hbar}{2m\omega}(a_+^2 + a_-^2 + 2N + 1)$$

$$\langle X^2 \rangle = \frac{e^{iE_0t/\hbar}\langle 0| + e^{-iE_1t/\hbar}\langle 1|}{\sqrt{2}} \frac{\hbar}{2m\omega}$$

$$\times (a_+^2 + a_-^2 + 2N + 1) \frac{e^{-iE_0t/\hbar}|0\rangle + e^{-iE_1t/\hbar}|1\rangle}{\sqrt{2}} \ (=)$$

$$a_+^2|0\rangle = \sqrt{2}|2\rangle$$

$$a_-^2|0\rangle = 0$$

$$a_+^2|1\rangle = \sqrt{2\times 3}|3\rangle = \sqrt{6}|3\rangle$$

$$a_-^2|1\rangle = 0$$

$$(=) \frac{\hbar}{2m\omega}\frac{1}{2}\left(e^{iE_0t/\hbar}\langle 0| + e^{iE_1t/\hbar}\langle 1|\right)$$

$$\times \Big(e^{-iE_0t/\hbar}\sqrt{2}|2\rangle + e^{-iE_0t/\hbar}|0\rangle$$

$$+ e^{-iE_1t/\hbar}\sqrt{6}|3\rangle + 3e^{-iE_1t/\hbar}|1\rangle\Big)$$

$$= \frac{\hbar}{2m\omega}\frac{1}{2}(1+3) = \frac{\hbar}{2m\omega}2$$

(9)

$$\therefore \langle \Delta X^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$$

$$\boxed{\langle \Delta X^2 \rangle = \frac{\hbar}{2m\omega} \left(2 - \cos^2 \frac{\Delta_{01} t}{\hbar}\right)}$$

$$\Delta_{01} := E_0 - E_1$$

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$$\begin{cases} J_\pm := a_\pm^\dagger a_\mp \hbar \\ J_z := \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) \\ N := a_+^\dagger a_+ + a_-^\dagger a_- \end{cases}$$

Known: $[a_\pm^\dagger, a_\pm] = -1$, $[a_\pm, a_\pm^\dagger] = +1$

$[a_\pm, a_\mp] = 0 \because$ assumed indep. (also for combos that contain †)

Let

$$N_\pm := a_\pm^\dagger a_\pm$$

$$[J_z, J_+] = \left[\frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \hbar a_+^\dagger a_-\right]$$

$$= \frac{\hbar^2}{2}\left([a_+^\dagger a_+, a_+^\dagger a_-] - [a_-^\dagger a_-, a_+^\dagger a_-]\right)$$

$$= \frac{\hbar^2}{2}\left(\underbrace{[a_+^\dagger a_+, a_+^\dagger]}_{a_+^\dagger \underbrace{[a_+, a_+^\dagger]}_{+1}} a_- - a_+^\dagger \underbrace{[a_-^\dagger a_-, a_-]}_{\underbrace{[a_-^\dagger, a_-]}_{-1} a_-}\right)$$

(10)

$$= \frac{\hbar^2}{2}\left(a_+^\dagger a_- + a_+^\dagger a_-\right)$$

$$= \hbar^2 a_+^\dagger a_-$$

$$\therefore \boxed{[J_z, J_+] = \hbar J_+}$$

$$[J_z, J_-] = \left[\frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \hbar a_-^\dagger a_+\right]$$

$$= \frac{\hbar^2}{2}\left([a_+^\dagger a_+, a_-^\dagger a_+] - [a_-^\dagger a_-, a_-^\dagger a_+]\right)$$

$$= \frac{\hbar^2}{2}\left(a_-^\dagger \underbrace{[a_+^\dagger a_+, a_+]}_{\underbrace{[a_+^\dagger, a_+]}_{-1} a_+} - \underbrace{[a_-^\dagger a_-, a_-^\dagger]}_{a_-^\dagger \underbrace{[a_-, a_-^\dagger]}_{+1}} a_+\right)$$

$$= \frac{\hbar^2}{2}\left(-a_-^\dagger a_+ - a_-^\dagger a_+\right)$$

$$= -\hbar^2 a_-^\dagger a_+$$

$$\therefore \boxed{[J_z, J_-] = -\hbar J_-}$$

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

$$J_\pm := J_x \pm i J_y$$

$$\left.\begin{aligned} J_+ &= J_x + i J_y \\ J_- &= J_x - i J_y \end{aligned}\right\} \quad J_x = \frac{J_+ + J_-}{\sqrt{2}}, \quad J_y = \frac{J_+ - J_-}{2i}$$

(11)

$$J_x^2 = \frac{J_+^2 + J_-^2 + J_+J_- + J_-J_+}{4}$$

$$J_y^2 = \frac{-J_+^2 - J_-^2 + J_+J_- + J_-J_+}{4}$$

$+$

$$J_x^2 + J_y^2 = \frac{1}{2}(J_+J_- + J_-J_+)$$

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

$$[J^2, J_z] = \left[\frac{1}{2}(J_+J_- + J_-J_+) + J_z^2, J_z\right]$$

$$= \frac{1}{2}\left([J_+J_-, J_z] + [J_-J_+, J_z]\right)$$

$$= \frac{1}{2}\Big(J_+[J_-, J_z] + [J_+, J_z]J_- + J_-[J_+, J_z] + [J_-, J_z]J_+\Big)$$

$$= \frac{1}{2}\Big(J_+(-)(-\hbar J_-) + (-)(\hbar J_+)J_- + J_-(-)(\hbar J_+) + (-)(-\hbar J_-)J_+\Big)$$

$$= \frac{\hbar}{2}(J_+J_- - J_+J_- - J_-J_+ + J_-J_+)$$

$$\therefore \boxed{[J^2, J_z] = 0}$$

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

$$= \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$$

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$$= \frac{1}{2}\left(\hbar a_+^\dagger a_- \hbar a_-^\dagger a_+ + \hbar a_-^\dagger a_+ \hbar a_+^\dagger a_-\right) + \frac{\hbar^2}{4}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)^2$$

$$= \frac{\hbar^2}{2}\left(a_- a_+^\dagger a_+ a_-^\dagger + a_+ a_-^\dagger a_- a_+^\dagger\right)$$
$$+ \frac{\hbar^2}{4}\left(a_+^\dagger a_+ a_+^\dagger a_+ + a_-^\dagger a_- a_-^\dagger a_- - a_+^\dagger a_+ a_-^\dagger a_- - a_-^\dagger a_- a_+^\dagger a_+\right) =$$

$$[a, a^\dagger] = 1 \Rightarrow a a^\dagger = 1 + N$$

$$= \frac{\hbar^2}{2}\left(N_+(1+N_-) + N_-(1+N_+)\right)$$
$$+ \frac{\hbar^2}{4}\left(N_+^2 + N_-^2 - 2N_+N_-\right)$$

$$= \frac{\hbar^2}{2}\left(N_+ + N_+N_- + N_- + \cancel{N_-N_+} + \frac{1}{2}N_+^2 + \frac{1}{2}N_-^2 - \cancel{N_+N_-}\right)$$

$$= \frac{\hbar^2}{2}\left(N + \frac{1}{2}(N_+ + N_-)^2\right)$$

$$= \frac{\hbar^2}{2}\left(N + \frac{1}{2}N^2\right)$$

$$\therefore \boxed{J^2 = \frac{\hbar^2}{2} N\left(\frac{N}{2} + 1\right)}$$

2.39

(a)

$$\Pi_x := p_x - \frac{eA_x}{c}$$

$$\Pi_y := p_y - \frac{eA_y}{c}$$

$$[\Pi_x, \Pi_y] = [p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y]$$

$$= -\frac{e}{c}\left([p_x, A_y] + [A_x, p_y]\right)$$

$$= -\frac{e}{c}\left([p_x, A_y] - [p_y, A_x]\right) \circledast$$

$$[p_i, f(\vec{x})] = p_i f(\vec{x}) - f(\vec{x}) p_i$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial x_i} f(\vec{x}) - f(\vec{x}) \frac{\hbar}{i}\frac{\partial}{\partial x_i}$$

$$= \frac{\hbar}{i}\frac{\partial f}{\partial x_i} + f(\vec{x})\frac{\hbar}{i}\frac{\partial}{\partial x_i} - f(\vec{x})\frac{\hbar}{i}\frac{\partial}{\partial x_i}$$

$$= \frac{\hbar}{i}\frac{\partial f}{\partial x_i}$$

$$\circledast\ -\frac{e}{c}\frac{\hbar}{i}\left(\partial_x A_y - \partial_y A_x\right), \quad \vec{B} = B\hat{z}$$

$$\therefore \boxed{[\Pi_x, \Pi_y] = \frac{ie\hbar B}{c}}$$

(14)

(b) $$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$$

$$= \frac{1}{2m}\vec{\Pi}^2 \qquad \left(\Pi_z = p_z \; \because A_z = 0 \; \because \vec{B} = \vec{\nabla}\times\vec{A}\right)$$

$$= \frac{1}{2m}p_z^2 + \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2\right)$$

$$y := \frac{c}{eB}\Pi_x \quad : \quad [y, \Pi_y] = i\hbar$$

$$H = \underset{\substack{\downarrow \\ \text{free Ham.}}}{\left\{\frac{1}{2m}p_z^2\right\}} + \underbrace{\left\{\frac{1}{2m}\Pi_y^2 + \frac{1}{2}m\frac{e^2B^2}{m^2c^2}y^2\right\}}_{\text{SHO Ham.}}$$

"If it looks like a duck,
if it sounds like a duck,
it's probably a duck."

$$\therefore \boxed{E = \frac{\hbar^2 k^2}{2m} + \hbar\frac{|e|B}{mc}\left(n + \frac{1}{2}\right)}$$

(15)

EXTRA 3

(a) Kermack - McCrea thm: (see proof at the end)

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^{A} e^{B} = e^{\frac{1}{2}[A,B]} e^{B} e^{A}$$

if $[A,[A,B]] = 0 = [B,[A,B]]$.

$$[a, a^{+}] = 1$$

$$\therefore [a,[a,a^{+}]] = 0 = [a^{+},[a,a^{+}]]$$

$$\therefore \Delta(\lambda) = e^{\lambda a^{+} - \lambda^{*} a}$$

$$= e^{-\frac{1}{2}[\lambda a^{+}, -\lambda^{*} a]} e^{\lambda a^{+}} e^{-\lambda^{*} a}$$

$$= e^{\frac{1}{2}|\lambda|^{2}[a^{+},a]} e^{\lambda a^{+}} e^{-\lambda^{*} a}$$

$$= e^{-|\lambda|^{2}/2} e^{\lambda a^{+}} e^{-\lambda^{*} a}$$

$$\Delta(\lambda)|0\rangle = e^{-|\lambda|^{2}/2} e^{\lambda a^{+}} e^{-\lambda^{*} a} |0\rangle \Leftrightarrow$$

$$a^{n}|0\rangle = \begin{cases} 0, & n \in \mathbb{N}^{+} \\ 1|0\rangle, & n = 0 \end{cases} \Rightarrow e^{-\lambda^{*} a}|0\rangle = |0\rangle$$

$$\Leftrightarrow e^{-|\lambda|^{2}/2} e^{\lambda a^{+}} |0\rangle$$

$$a^{+}|0\rangle = \sqrt{1}\,|1\rangle$$

$$a^{+2}|0\rangle = \sqrt{1\times 2}\,|2\rangle$$

$$\dots$$

$$a^{+n}|0\rangle = \sqrt{n!}\,|n\rangle$$

$$\therefore e^{\lambda a^{+}}|0\rangle = \sum_{n\geq 0} \frac{(\lambda a^{+})^{n}}{n!}|0\rangle = \sum_{n\geq 0} \frac{\lambda^{n}}{n!}\sqrt{n!}\,|n\rangle$$

$$= \sum_{n\geq 0} \frac{\lambda^{n}}{\sqrt{n!}}|n\rangle$$

$$\therefore\ \Delta(\lambda)\,|0\rangle = e^{-|\lambda|^2/2} \sum_{n\geq 0} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

Now let's see if this is an eigenstate of a:

$$a\,(\Delta(\lambda)|0\rangle) = e^{-|\lambda|^2/2} \sum_{n\geq 0} \frac{\lambda^n}{\sqrt{n!}}\, a|n\rangle \ominus$$

$$a|n\rangle = a\,\frac{a^{+n}}{\sqrt{n!}}\,|0\rangle$$

$$= \frac{a^{+n}a + [a, a^{+n}]}{\sqrt{n!}}\,|0\rangle$$

$$[a, a^+] = 1$$

$$[a, a^{+2}] = a^+[a,a^+] + [a,a^+]a^+ = 2a^+$$

$$\dots$$

$$[a, a^{+n}] = n a^{+\,n-1} = \frac{\partial}{\partial a^+} a^{+n}$$

$$\therefore\ a|n\rangle = \frac{1}{\sqrt{n!}} \frac{\partial}{\partial a^+} a^{+n} |0\rangle$$

$$\therefore\ a\,(\Delta(\lambda)|0\rangle) = e^{-|\lambda|^2/2} \sum_{n\geq 0} \frac{\lambda^n}{\sqrt{n!}} \frac{1}{\sqrt{n!}} \frac{\partial}{\partial a^+} a^{+n} |0\rangle$$

$$= e^{-|\lambda|^2/2} \frac{\partial}{\partial a^+} \sum_{n\geq 0} \frac{(\lambda a^+)^n}{n!} |0\rangle$$

$$= \frac{\partial}{\partial a^+}\left(e^{-|\lambda|^2/2} e^{\lambda a^+}\right) |0\rangle$$

$$= \lambda e^{-|\lambda|^2/2} e^{\lambda a^+} |0\rangle$$

$$= \lambda e^{-|\lambda|^2/2} e^{\lambda a^+} e^{\lambda^* a} |0\rangle$$

$$= \lambda\,(\Delta(\lambda)|0\rangle)$$

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$\therefore \Delta(\lambda)|0\rangle$ is the eigenstate of a w/ eigenvalue λ.

$\therefore |\lambda\rangle = \Delta(\lambda)|0\rangle$ qed

(b) $Me^{L}M^{-1} = M\left(1 + L + \frac{1}{2!}L^2 + \dots\right)M^{-1}$

$$= 1 + MLM^{-1} + \frac{1}{2!}ML^2M^{-1} + \dots$$

$$= 1 + MLM^{-1} + \frac{1}{2!}MLM^{-1}MLM^{-1} + \dots$$

$$= 1 + (MLM^{-1}) + \frac{1}{2!}(MLM^{-1})^2 + \dots$$

$$= e^{MLM^{-1}}$$ qed

$$\mathcal{U}_0(t)^\dagger \Delta(\lambda)\mathcal{U}_0(t) = e^{iH_0t/\hbar} e^{\lambda a^\dagger - \lambda^* a} e^{-iH_0t/\hbar}$$

$$= e^{e^{iH_0t/\hbar}(\lambda a^\dagger - \lambda^* a)e^{-iH_0t/\hbar}}$$

$$= e^{\lambda a^\dagger(t) - \lambda^* a(t)}$$

$$H_0 = \hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right)$$

$$[a, H_0] = \hbar\omega_0[a, a^\dagger a] = \hbar\omega_0 \underbrace{[a, a^\dagger]}_{1} a = \hbar\omega_0 a$$

$$\therefore \dot{a} = \frac{\hbar\omega_0 a}{i\hbar} \Rightarrow a(t) = a e^{-i\omega_0 t}$$

$$[a^\dagger, H_0] = \hbar\omega_0[a^\dagger, a^\dagger a] = \hbar\omega_0 a^\dagger \underbrace{[a^\dagger, a]}_{-1} = -\hbar\omega_0 a^\dagger$$

$$\therefore \dot{a}^\dagger = \frac{-\hbar\omega_0}{i\hbar}a^\dagger \Rightarrow a(t)^\dagger = a^\dagger e^{i\omega_0 t}$$

$$\therefore \boxed{\mathcal{U}_0(t)^\dagger \Delta(\lambda)\mathcal{U}_0(t) = e^{\lambda e^{i\omega_0 t}a^\dagger - \lambda^* e^{-i\omega_0 t}a}}$$

(18)

$$|\alpha(0)\rangle = |\lambda_0\rangle$$

$$|\alpha(t)\rangle = \mathcal{U}_0(t)|\lambda_0\rangle$$

$$= \mathcal{U}_0(t)\Delta(\lambda_0)|0\rangle$$

$$= \mathcal{U}_0(t)\Delta(\lambda_0)\mathcal{U}_0(t)^\dagger\mathcal{U}_0(t)|0\rangle$$

$$\mathcal{U}_0(t)\Delta(\lambda_0)\mathcal{U}_0(t)^\dagger = e^{-iH_0t/\hbar}\Delta(\lambda_0)e^{iH_0t/\hbar}$$

$$= e^{iH_0(-t)/\hbar}\Delta(\lambda_0)e^{-iH_0(-t)/\hbar}$$

$$= \mathcal{U}_0(t)^\dagger\Delta(\lambda)\mathcal{U}_0(t)\Big|_{t\to -t}$$

$$= e^{\lambda e^{-i\omega_0 t}a^\dagger - \lambda^* e^{i\omega_0 t}a}$$

$$\mathcal{U}_0(t)|0\rangle = e^{-iH_0t/\hbar}|0\rangle = e^{-iE_0t/\hbar}|0\rangle = e^{-i\omega_0 t/2}|0\rangle$$

$$\therefore |\alpha(t)\rangle = e^{\lambda e^{-i\omega_0 t}a^\dagger - \lambda^* e^{i\omega_0 t}a}e^{-i\omega_0 t/2}|0\rangle, \quad \lambda(t) := \lambda e^{-i\omega_0 t}$$

$$= e^{-i\omega_0 t/2}e^{\lambda(t)a^\dagger - \lambda^*(t)a}|0\rangle$$

$$= e^{-i\omega_0 t/2}\Delta(\lambda(t))|0\rangle$$

$$\boxed{|\alpha(t)\rangle = e^{-i\omega_0 t/2}|\lambda(t)\rangle}$$

(c) $H_0 = \dfrac{P^2}{2m} + \dfrac{1}{2}m\omega^2 X^2$

$$H_1 = -fX$$

$$\therefore H = H_0 + H_1 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X^2 - fX$$

$$[X,H] = \frac{1}{2m}[X,P^2] = \frac{i\hbar P}{m}$$

$$[P,H] = \frac{1}{2}m\omega^2[P,X^2] - f[P,X] = \frac{m\omega^2}{2}(-2i\hbar X) - f(-i\hbar)$$

$$= -i\hbar m\omega^2 X + i\hbar f$$

(19)

$$\therefore \dot{X} = \frac{1}{i\hbar}\frac{i\hbar P}{m} = \frac{P}{m}$$

$$\dot{P} = \frac{1}{i\hbar}\left(-i\hbar m\omega^2 X + i\hbar f\right)$$

$$= -m\omega^2 X + f$$

$$\therefore \ddot{X} = \frac{\dot{P}}{m} = -\omega^2 X + \frac{f}{m}$$

$$\boxed{\ddot{X} + \omega^2 X = \frac{f}{m}}$$

(d) $$|\alpha(t)\rangle = e^{-iH_0t/\hbar}\,|\alpha_I(t)\rangle$$

$$\therefore |\alpha_I(t)\rangle = e^{iH_0t/\hbar}\,|\alpha(t)\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\alpha_I(t)\rangle = i\hbar\,\frac{iH_0}{\hbar}\,e^{iH_0t/\hbar}\,|\alpha(t)\rangle + e^{iH_0t/\hbar}\,\underbrace{i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle}_{H|\alpha(t)\rangle}$$

↳ total hamiltonian

$$= e^{iH_0t/\hbar}\,(-H_0)\,|\alpha(t)\rangle + e^{iH_0t/\hbar}\,H\,|\alpha(t)\rangle$$

$$= e^{iH_0t/\hbar}\,H_1\,|\alpha(t)\rangle$$

$$= e^{iH_0t/\hbar}\,H_1\,e^{-iH_0t/\hbar}\,|\alpha_I(t)\rangle$$

$$= H_I(t)\,|\alpha_I(t)\rangle \quad \text{qed}$$

$$H_I(t) := e^{iH_0t/\hbar}\,H_1\,e^{-iH_0t/\hbar}$$

$$= e^{iH_0t/\hbar}\,(-fX)\,e^{-iH_0t/\hbar}$$

$$= -f\,e^{iH_0t/\hbar}\,X\,e^{-iH_0t/\hbar}$$

$$= -f\,e^{iH_0t/\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\,(a^\dagger + a)\,e^{-iH_0t/\hbar}$$

$$= -f \underbrace{\sqrt{\frac{\hbar}{2m\omega_0}}}_{=:\,x_0} \Big(\underbrace{e^{iH_0t/\hbar} a^\dagger e^{-iH_0t/\hbar}}_{a^\dagger(t)} + \underbrace{e^{iH_0t/\hbar} a e^{-iH_0t/\hbar}}_{a(t)} \Big)$$

$$= -f x_0 \left(a^\dagger(t) e^{i\omega_0 t} + a e^{i\omega_0 t} \right) \quad \text{from earlier}$$

$$= \left(-f x_0 e^{i\omega_0 t}\right) a^\dagger + \left(-f x_0 e^{-i\omega_0 t}\right) a$$

$$= g(t) a^\dagger + g(t)^* a$$

$$\therefore \boxed{g(t) = -f x_0 e^{i\omega_0 t}} \qquad \text{where } x_0 := \sqrt{\frac{\hbar}{2m\omega_0}}$$

But let's compute $e^{iH_0t/\hbar} a^\dagger e^{-iH_0t/\hbar}$ and $e^{iH_0t/\hbar} a e^{-iH_0t/\hbar}$ by using the Baker-Campbell-Hausdorff formula:

$$e^{iG\lambda} A e^{-iG\lambda} = A + i\lambda [G, A] + \frac{(i\lambda)^2}{2!} [G, [G, A]] + \dots$$

Put $G = H_0$, $\lambda = t/\hbar$.

$$H_0 = \hbar\omega_0 \left(a^\dagger a + \tfrac{1}{2}\right)$$

$$[H_0, a] = \hbar\omega_0 [a^\dagger a, a] = \hbar\omega_0 [a^\dagger, a] a = -\hbar\omega_0 a$$

$$[H_0, [H_0, a]] = [H_0, -\hbar\omega_0 a] = (-\hbar\omega_0)^2 a$$

$$\dots$$

$$[H_0, a^\dagger] = \hbar\omega_0 [a^\dagger a, a^\dagger] = \hbar\omega_0 a^\dagger [a, a^\dagger] = \hbar\omega_0 a^\dagger$$

$$[H_0, [H_0, a^\dagger]] = [H_0, \hbar\omega_0 a^\dagger] = (\hbar\omega_0)^2 a^\dagger$$

$$\dots$$

$$\therefore e^{iH_0t/\hbar} a^\dagger e^{-iH_0t/\hbar} = a^\dagger + \left(\frac{it}{\hbar}\right)(\hbar\omega_0) a^\dagger + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 (\hbar\omega_0)^2 a^\dagger + \dots$$

$$= a^\dagger \left(1 + (i\omega_0 t) + \frac{1}{2!}(i\omega_0 t)^2 + \dots \right)$$

$$= a^\dagger e^{i\omega_0 t} \checkmark$$

(21)

$$e^{iH_0t/\hbar} a e^{-iH_0t/\hbar} = a + \left(\frac{it}{\hbar}\right)(-\hbar\omega_0)a + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2(-\hbar\omega)^2 a + \dots$$

$$= a\left(1 + (-i\omega_0 t) + \frac{1}{2!}(-i\omega_0 t)^2 + \dots\right)$$

$$= a e^{-i\omega_0 t/\hbar}$$

(e) $i\hbar \frac{\partial}{\partial t} \mathcal{U}_I(t) = H_I(t)\mathcal{U}_I(t)$

$\mathcal{U}_I(t) = e^{h(t)a^\dagger - h(t)^* a} e^{i\beta(t)}$ this is of the form $\Delta(h(t))$

$= e^{-|h(t)|^2/2} e^{h(t)a^\dagger} e^{-h(t)^* a} e^{i\beta(t)}$

$$\frac{\partial \mathcal{U}_I(t)}{\partial t} = -\frac{\partial |h(t)|^2}{\partial t} e^{-|h(t)|^2/2} e^{h(t)a^\dagger} e^{-h(t)^* a} e^{i\beta(t)}$$

$$+ e^{-|h(t)|^2/2} \dot{h}(t)a^\dagger e^{h(t)a^\dagger} e^{-h(t)^* a} e^{i\beta(t)}$$

$$+ e^{-|h(t)|^2/2} e^{h(t)a^\dagger} (-\dot{h}^*(t)a) e^{-h(t)^* a} e^{i\beta(t)}$$

$$+ e^{-|h(t)|^2/2} e^{h(t)a^\dagger} e^{-h(t)^* a} i\dot{\beta}(t) e^{i\beta(t)}$$

$$= \left(-\frac{\partial |h(t)|^2}{\partial t} + \dot{h}(t)a^\dagger + i\dot{\beta}(t)\right)\mathcal{U}_I(t)$$

$$- \dot{h}^*(t) e^{-|h(t)|^2/2} \underbrace{e^{h(t)a^\dagger} a}_{a e^{h(t)a^\dagger} + [e^{h(t)a^\dagger}, a]} e^{-h(t)^* a} e^{i\beta(t)}$$

$$= \left(-\frac{\partial |h(t)|^2}{\partial t} + \dot{h}(t)a^\dagger + i\dot{\beta}(t) - \dot{h}(t)^* a\right)\mathcal{U}_I(t)$$

$$- \dot{h}^*(t) e^{-|h(t)|^2/2} [e^{h(t)a^\dagger}, a] e^{-h(t)^* a} e^{i\beta(t)} \quad (=)$$

$[a, a^{\dagger n}] = \frac{\partial}{\partial a^\dagger} a^{\dagger n}$ from earlier

$\therefore [a, e^{\lambda a^\dagger}] = \sum_n c_n \lambda^n \frac{\partial}{\partial a^\dagger} a^{\dagger n} = \frac{\partial}{\partial a^\dagger} e^{\lambda a^\dagger} = \lambda e^{\lambda a^\dagger}$

$$\stackrel{?}{=} \left(-\frac{\partial |h(t)|^2}{\partial t} + \dot{h}(t)a^\dagger - \dot{h}^*(t)a + i\dot{\beta}(t)\right)\mathcal{U}_I(t)$$

$$-\dot{h}^*(t)e^{-|h(t)|^2/2}(-h(t)e^{h(t)a^\dagger})e^{-h(t)^*a}e^{i\beta(t)}$$

$$= \left(-\frac{\partial |h(t)|^2}{\partial t} + \dot{h}(t)a^\dagger - \dot{h}^*(t)a + i\dot{\beta}(t) + h(t)\dot{h}^*(t)\right)\mathcal{U}_I(t)$$

$$= \frac{H_I(t)}{i\hbar}\mathcal{U}_I(t)$$

$$= \frac{g(t)a^\dagger + g(t)^* a}{i\hbar}\mathcal{U}_I(t)$$

Since a, $a^\dagger$, and 1 are linearly indep., we directly have

$$\boxed{\begin{aligned} &\dot{h}(t) = \frac{g(t)}{i\hbar} \\ &-\frac{\partial |h(t)|^2}{\partial t} + i\dot{\beta}(t) + h(t)\dot{h}^*(t) = 0 \end{aligned}}$$

$g(t) = -fx_0 e^{i\omega_0 t}$, $x_0 = \sqrt{\frac{\hbar}{2m\omega_0}}$

by just equating the coefficients.

(f) $|\alpha_I(0)\rangle = |\lambda_0\rangle$

$$|\alpha_I(t)\rangle = \mathcal{U}_I(t)|\lambda_0\rangle$$

but notice that $\mathcal{U}_I(t) = e^{i\beta(t)}\Delta(h(t))$:

$$|\alpha_I(t)\rangle = e^{i\beta(t)}\Delta(h(t))\Delta(\lambda_0)|0\rangle$$

What to do w/ this operators?

Since Δ is a displacement operator, by intuition $\Delta(\lambda)\Delta(\lambda')$ should be related to $\Delta(\lambda+\lambda')$:

$$\Delta(\lambda+\lambda') = e^{(\lambda+\lambda')a^\dagger - (\lambda^*+\lambda'^*)a}$$

$$= e^{(\lambda a^\dagger - \lambda^* a) + (\lambda' a^\dagger - \lambda'^* a)}$$

$$[\lambda a^\dagger - \lambda^* a, \lambda' a^\dagger - \lambda'^* a] = -\lambda\lambda'^* \underbrace{[a^\dagger, a]}_{-1} - \lambda^*\lambda' \underbrace{[a, a^\dagger]}_{1}$$

$=0$ $=0$

$$= \lambda\lambda'^* - \lambda^*\lambda'$$

$$= 2i\, \mathrm{Im}\, \lambda\lambda'^* \quad \text{indep of } a \text{ and } a^\dagger$$

$$\therefore [\lambda a^\dagger - \lambda^* a, [\lambda a^\dagger - \lambda^* a, \lambda' a^\dagger - \lambda'^* a]] = 0$$

$$= [\lambda' a^\dagger - \lambda'^* a, [\lambda a^\dagger - \lambda^* a, \lambda' a^\dagger - \lambda'^* a]]$$

$$\therefore \Delta(\lambda + \lambda') = e^{-\frac{1}{2}[\lambda a^\dagger - \lambda^* a, \lambda' a^\dagger - \lambda'^* a]} e^{\lambda a^\dagger - \lambda^* a} e^{\lambda' a^\dagger - \lambda'^* a}$$

$$= e^{-\frac{1}{2} 2i\, \mathrm{Im}\, \lambda\lambda'^*} \Delta(\lambda)\Delta(\lambda')$$

$$\therefore \Delta(\lambda)\Delta(\lambda') = e^{i\, \mathrm{Im}\, \lambda\lambda'^*} \Delta(\lambda + \lambda')$$

$$\therefore \Delta(h(t))\Delta(\lambda_0) = e^{i\, \mathrm{Im}\, h(t)\lambda_0^*} \Delta(h(t) + \lambda_0)$$

$$\therefore |\alpha_I(t)\rangle = e^{i\beta(t)} e^{i\, \mathrm{Im}\, h(t)\lambda_0^*} \Delta(h(t) + \lambda_0)|0\rangle$$

$$= e^{i(\beta + \mathrm{Im}\, h(t)\lambda_0^*)} |h(t) + \lambda_0\rangle$$

$$|\alpha(t)\rangle = e^{-iH_0 t/\hbar} |\alpha_I(t)\rangle$$

$$= e^{i(\beta + \mathrm{Im}\, h(t)\lambda_0^*)} \mathcal{U}_0(t)\Delta(h(t) + \lambda_0)|0\rangle$$

$$= e^{i(\beta + \mathrm{Im}\, h(t)\lambda_0^*)} \underbrace{\mathcal{U}_0(t)\Delta(\lambda')\mathcal{U}_0(t)^\dagger}_{e^{\lambda' e^{-i\omega_0 t} a^\dagger - \lambda'^* e^{i\omega_0 t} a}} \underbrace{\mathcal{U}_0(t)|0\rangle}_{e^{-iE_0 t/\hbar}|0\rangle}, \quad \lambda' := h(t) + \lambda_0$$

$$= e^{i(\beta + \mathrm{Im}\, h(t)\lambda_0^*)} \Delta(\lambda' e^{-i\omega_0 t})|0\rangle$$

$$= e^{i\gamma(t)} |\lambda(t)\rangle$$

where $\lambda(t) = \lambda' e^{-i\omega_0 t}$

$$\boxed{\lambda(t) = (h(t) + \lambda_0)\, e^{-i\omega_0 t}}$$

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$$\dot{h}(t) = \frac{g(t)}{i\hbar} = \frac{1}{i\hbar}\left(-f x_0 e^{i\omega_0 t}\right)$$

$$\therefore h(t) = \frac{-f x_0}{i\hbar}\,\frac{e^{i\omega_0 t}}{i\omega_0}$$ up to some additive const.

$$\boxed{h(t) = \frac{f x_0}{\hbar\omega_0} e^{i\omega_0 t}}$$

$$\lambda(t) = \left(\frac{f x_0}{\hbar\omega_0} e^{i\omega_0 t} + \lambda_0\right) e^{-i\omega_0 t}$$

$$= \frac{f x_0}{\hbar\omega_0} + \lambda_0 e^{-i\omega_0 t}$$

Since λ_0 is a complex number, we can write it as

$$\lambda_0 = A + iB$$

Then

$$\lambda(t) = \frac{f x_0}{\hbar\omega_0} + (A+iB)(\cos\omega_0 t - i\sin\omega_0 t)$$

$$= \underbrace{\frac{f x_0}{\hbar\omega_0} + A\cos\omega_0 t + B\sin\omega_0 t}_{\text{real}} + i(\cdots)$$

$$x(t) := \sqrt{\frac{2\hbar}{m\omega_0}}\ \mathrm{Re}\,\lambda(t) = 2\sqrt{\frac{\hbar}{2m\omega_0}}\ \mathrm{Re}\,\lambda(t)$$

$$= 2x_0\left(\frac{f x_0}{\hbar\omega_0} + A\cos\omega_0 t + B\sin\omega_0 t\right)$$

$$= f\,\underbrace{\frac{2x_0^2}{\hbar\omega_0}}_{\frac{2}{\hbar\omega_0}\,\frac{\hbar}{2m\omega_0}} + A'\cos\omega_0 t + B'\sin\omega_0 t$$

$$= \frac{f}{m\omega_0^2} + A'\cos\omega_0 t + B'\sin\omega_0 t$$

Recall the equation of motion:

$$\ddot{x} + \omega_0^2 x = \frac{f}{m}$$

From Phys209, we know that $\exists$ two types of solutions here:

$$\ddot{x}_c + \omega_0^2 x_c = 0 \quad \text{complementary solution} \leftrightarrow \text{homogeneous eqn}$$

$$\ddot{x}_p + \omega_0^2 x_p = \frac{f}{m} \quad \text{particular solution} \leftrightarrow \text{inhomo. eqn.}$$

The complementary solution is apparently

$$x_c(t) = C\cos\omega_0 t + D\sin\omega_0 t$$

Since the force is const, so should be the particular solution:

$$x_p = K, \quad \dot{x}_p = \ddot{x}_p = 0$$

$$\omega_0^2 K = \frac{f}{m} \quad \therefore K = \frac{f}{m\omega_0^2}$$

$$\therefore x(t) = \frac{f}{m\omega_0^2} + C\cos\omega_0 t + D\sin\omega_0 t$$

which is exactly $\sqrt{2\hbar/m\omega_0}\,\mathrm{Re}\,\lambda(t)$.

EXTRA 3 (Proof of the thm)

(a) <u>Kermack - McCrea thm</u> For $[A,[A,B]] = 0 = [B,[A,B]]$,

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^{A} e^{B} \qquad \text{"AB ordered"}$$

$$= e^{-\frac{1}{2}[A,B]} e^{B} e^{A} \qquad \text{"BA ordered"}$$

Proof. Let a and b denote the 'scaler' operators that correspond to A and B, resp., that is, $ab = ba$. Let

$$f(a,b)_{AB} = f(a \to A, b \to B) \text{ in the order } AB$$

For ex, if we have $f(a,b) = e^{a+b} = e^{a}e^{b}$, then

$$f(a,b)_{AB} = e^{A}e^{B} \quad \text{et} \quad f(a,b)_{BA} = e^{B}e^{A}$$

Let

$$F := e^{A+B}$$

which we also want to be equal to $f(a,b)_{AB}$.
Then we see that

$$\frac{\partial F}{\partial A} = \frac{\partial F}{\partial B} = F$$

$$\therefore \frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = f \qquad \therefore f(a,b) = C\, e^{a+b}$$

This means that C should satisfy

$$Ce^{A}e^{B} = e^{A+B}$$

where we tacitly assumed that C commutes w/ both A and B.

The standard trick in dealing w/ the algebra of nested commutators — as we have learned in Baker-Hausdorff formula — is to parametrize the exponentials:

$$C(\lambda)e^{\lambda A}e^{\lambda B} = e^{\lambda(A+B)} \quad \Big| \leftarrow e^{-\lambda B}$$

$$C(\lambda)e^{\lambda A} = e^{\lambda(A+B)}e^{-\lambda B} \quad \Big| \leftarrow e^{-\lambda A}$$

$$C(\lambda) = e^{\lambda(A+B)}e^{-\lambda B}e^{-\lambda A}$$

$$C(0) = 1$$

$$\frac{dC(\lambda)}{d\lambda} = (A+B)e^{\lambda(A+B)}e^{-\lambda B}e^{-\lambda A}$$
$$+ e^{\lambda(A+B)}(-B)e^{-\lambda B}e^{-\lambda A}$$
$$+ e^{\lambda(A+B)}e^{-\lambda B}(-A)e^{-\lambda A}, \qquad [e^{\lambda(A+B)}, (A+B)] = 0$$

$$= e^{\lambda(A+B)}(A+B)e^{-\lambda B}e^{-\lambda A}$$
$$- e^{\lambda(A+B)}Be^{-\lambda B}e^{-\lambda A}$$
$$- e^{-\lambda(A+B)}e^{-\lambda B}Ae^{-\lambda A}$$

$$= e^{\lambda(A+B)}[A, e^{-\lambda B}]e^{-\lambda A}$$

$$[A, e^{-\lambda B}] = \sum_n c_n(-\lambda)^n [A, B^n]$$

$$[A, B^2] = B[A,B] + [A,B]B = 2B[A,B] \quad \text{(recall assumption)}$$

$$[A, B^3] = B\underbrace{[A, B^2]}_{2B[A,B]} + [A,B]B^2 = 3B^2[A,B]$$

$$[A, e^{-\lambda B}] = \sum_n c_n (-\lambda)^n n B^{n-1} [A, B]$$

$$= [A, B] \sum_n c_n (-\lambda)^n n B^{n-1}$$

$$= [A, B] \frac{\partial}{\partial B} \sum_n c_n (-\lambda)^n B^n$$

$$= [A, B] \frac{\partial}{\partial B} e^{-\lambda B}$$

$$= -\lambda [A, B] e^{-\lambda B}$$

$$= -\lambda e^{-\lambda B} [A, B]$$

$$\therefore \frac{dC(\lambda)}{d\lambda} = e^{\lambda(A+B)} \left(-\lambda e^{-\lambda B} [A, B]\right) e^{-\lambda A}$$

$$= -\lambda [A, B] e^{\lambda(A+B)} e^{-\lambda B} e^{-\lambda A}$$

$$= -\lambda [A, B] C(\lambda)$$

Since C commutes w/ A and B, it also commutes w/ $[A, B]$, so we can easily integrate it.

$$\frac{dC(\lambda)}{C(\lambda)} = -\lambda [A, B] d\lambda$$

$$\ln C(\lambda) = -\frac{\lambda^2}{2} [A, B] + \ln C(0)$$

$$C(\lambda) = \underbrace{C(0)}_{1} e^{-\frac{\lambda^2}{2}[A, B]}$$

$$= e^{-\frac{\lambda^2}{2}[A, B]}$$

Finally, by taking $\lambda = 1$,

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^{A} e^{B}$$

and since $A \leftrightarrow B$ implies $[A,B] \longrightarrow -[A,B]$,

$$e^{A+B} = e^{\frac{1}{2}[A,B]} e^{B} e^{A} \qquad \text{qed.}$$

①

EXTRA 4

(a) $H_0 = \frac{\vec{L}^2}{2I}$

$\vec{L}^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle$

$\psi_{lm}(\vec{x}) = \langle \hat{n} | lm\rangle = \langle \theta\varphi | lm\rangle =: Y_{lm}(\theta, \varphi)$

$E_l = \frac{\hbar^2 l(l+1)}{2I}$

$\exists$ 2 quantum #s that describe the dynamics, l and m:

$l \in \mathbb{N}$

$m = -l : l \Rightarrow 2l+1$ values $\forall l$

Notice that for all l, $\exists$ $(2l+1)$-many m values but the energy eigenvalues are indep. of m, so the degeneracy is $2l+1$.

(b) $H_1 = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_2}$

$\exists$ two ways to proceed. I will do one of them:

$L_x^2 + L_y^2 = L^2 - L_z^2 \quad \left(\underset{\substack{\downarrow \\ 2^{nd} \text{ way}}}{=} \frac{1}{2}(L_+L_- + L_-L_+) \right)$

$H_1 = \frac{L^2}{2I} - \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_2}\right) L_z^2$

(2)

Since $[L^2, L_z^2] = 0$, and since $|lm\rangle$ is eigenstate of both of them, the spherical harmonics is still the eigenfunction. But the energy eigenvalues and the degeneracy need carlful examination:

$$L^2 \rightarrow \hbar^2 l(l+1)$$

$$L_z \rightarrow \hbar m$$

$$\therefore E_{lm} = \frac{\hbar^2 l(l+1)}{2I_1} - \frac{1}{2}\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\hbar^2 m^2$$

Again the dynamics depend on two quantum numbers, l and m, but the energy eigenvalues also depend on l and m explicitly.

Intuitively, we expect the greater the number of quantum #s the energy depends on, the smaller the degeneracy. Yes, $\exists$ still $(2l+1)$-many m values for each l, we have a different energy for each $|m|$. That is, the degeneracy is significantly lifted: we have 2 degeneracies left — $\pm m$.

③

EXTRA 5

3.5

- Method 1: Use explicit forms of S_x and S_z.
- Method 2:

Cayley-Hamilton thm: Any given square matrix satisfies its secular eqn.

Proof. (See your Math 260 notes.)

$\therefore$ Cayley-Hamilton thm says that

$$\prod_{i=1}^{N} (A - \lambda_i) = 0$$

where A is an $N \times N$ matrix and the λ_i are the eigenvalues of A.

Think about it.

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3.14

$$(G_i)_{jk} = -i\hbar\,\varepsilon_{ijk}$$

$$([G_i, G_j])_{mn} = (G_i G_j)_{mn} - (G_j G_i)_{mn}$$

$$= (G_i)_{mk}(G_j)_{kn} - (G_j)_{mk}(G_i)_{kn}$$

$$= (-i\hbar\,\varepsilon_{imk})(-i\hbar\,\varepsilon_{jkn}) - (-i\hbar\,\varepsilon_{jmk})(-i\hbar\,\varepsilon_{ikn})$$

$$= (-i\hbar)^2(\varepsilon_{imk}\varepsilon_{njk} - \varepsilon_{jmk}\varepsilon_{nik})$$

$$= (-i\hbar)^2\left((\delta_{in}\delta_{mj} - \delta_{ij}\delta_{mn}) - (\delta_{jn}\delta_{mi} - \delta_{ji}\delta_{mn})\right)$$

$$= (-i\hbar)^2\left(\delta_{in}\delta_{mj} - \cancel{\delta_{ij}\delta_{mn}} - \delta_{im}\delta_{jn} + \cancel{\delta_{ij}\delta_{mn}}\right)$$

$$= (-i\hbar)^2(\delta_{in}\delta_{mj} - \delta_{im}\delta_{jn})$$

$$= (-i\hbar)^2\,\varepsilon_{ijk}\underbrace{\varepsilon_{nmk}}_{-\varepsilon_{mnk} = -\varepsilon_{kmn}}$$ by trial and error

$$= i\hbar\,\varepsilon_{ijk}(-i\hbar\,\varepsilon_{kmn})$$

$$= i\hbar\,\varepsilon_{ijk}(G_k)_{mn}$$

$\therefore [G_i, G_j] = i\hbar\,\varepsilon_{ijk}G_k$ qed

5

Matrix representation of the J_i for $j=1$:

$$|11\rangle \doteq \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |10\rangle \doteq \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |1\,{-1}\rangle \doteq \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$J_z |jm\rangle = \hbar m |jm\rangle$$

$$\therefore \langle jm' | J_z | jm\rangle = \hbar m\, \delta_{m'm}$$

$$J_\pm |jm\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)}\, |j\, m\pm 1\rangle$$

$$\therefore \langle jm' | J_\pm | jm\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)}\; \delta_{m'\, m\pm 1}$$

$$\therefore (J_z)_{m'm} = \hbar m\, \delta_{m'm}$$

$$(J_+)_{m'm} = \hbar \sqrt{2 - m(m+1)}\; \delta_{m'\, m+1}$$

$$(J_-)_{m'm} = \hbar \sqrt{2 - m(m-1)}\; \delta_{m'\, m-1}$$

$$\therefore J_z \doteq \hbar \begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}$$

$$J_+ \doteq \hbar \begin{pmatrix}0 & \sqrt{2} & 0\\ & 0 & \sqrt{2}\\ & & 0\end{pmatrix}, \qquad J_- \doteq \hbar \begin{pmatrix}0 & & \\ \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0\end{pmatrix}$$

$$J_x = \frac{J_+ + J_-}{2} \doteq \frac{\hbar}{2}\begin{pmatrix}0 & \sqrt{2} & 0\\ \sqrt{2} & 0 & \sqrt{2}\\ 0 & \sqrt{2} & 0\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$J_y = \frac{J_+ - J_-}{2i} = \frac{\hbar}{2i}\begin{pmatrix}0 & \sqrt{2} & 0\\ -\sqrt{2} & 0 & \sqrt{2}\\ 0 & -\sqrt{2} & 0\end{pmatrix} = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$$

⑥

Matrix representation of G_i :

$$(G_1)_{\substack{jk\\23\\32}} = -i\hbar\,\varepsilon_{1\substack{jk\\23\\32}}$$

$$(G_2)_{\substack{jk\\31\\13}} = -i\hbar\,\varepsilon_{2\substack{jk\\31\\13}}$$

$$(G_3)_{\substack{jk\\12\\21}} = -i\hbar\,\varepsilon_{3\substack{jk\\12\\21}}$$

$$\therefore\ G_1 = -i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$G_2 = -i\hbar \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$G_3 = -i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since we are asked to relate G_i to J_i by a similarity transformation in the basis where J_3 is diagonal, if we can find the matrix that diagonalizes G_3, we are done.

Eigenvalues of G_3 :

$$\begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2+1) = 0$$

$$\therefore\ \lambda = -i\hbar\{0, \pm i\} = \hbar\{0, \pm 1\}$$ as expected $\therefore\ \mathcal{U}^{-1} G_3 \mathcal{U} = \underset{\text{diag.}}{\underset{\downarrow}{J_3}}$

⑦

Eigenvectors of G_3:

$$\begin{pmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda \neq 0: \quad \left.\begin{array}{l} c = 0 \\ -\lambda a + b = 0 \Rightarrow b = \lambda a \end{array}\right\} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$\lambda = 0: \quad \left.\begin{array}{l} c \neq 0 \ \text{(free)} \\ a = 0 \\ b = 0 \end{array}\right\} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

let $\hat{a}_\pm := \dfrac{\hat{x} \pm i\hat{y}}{\sqrt{2}}$

$\hat{a}_0 := \hat{z}$

where $\hat{x} \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{y} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\hat{z} := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

This will be useful later.

Now, let's consider the transformation $\vec{v} \to \vec{v} + \hat{n}\delta\varphi \times \vec{v}$ before the significance of $\mathcal{U}$, which looks like

$$\mathcal{U} = (\hat{a}_+ \ \hat{a}_0 \ \hat{a}_-) = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

(check $\mathcal{U}^\dagger G_3 \mathcal{U} = J_3$.)

(8)

Now,

$$v_i \to v_i + \varepsilon_{ijk}\,\hat{n}_j\,\delta\varphi\, v_k$$

$$\to v_i + \frac{(G_i)_{jk}}{-i\hbar}\,\hat{n}_j\,\delta\varphi\, v_k$$

$$\to v_i + \frac{i}{\hbar}\,\delta\varphi\,\underbrace{\hat{n}_j\, G^i_{jk}\, v_k}_{\hat{n}\cdot G^i\vec{v}}$$

(compare: $a_i b_j A_{ij} = \vec{a}\cdot A\vec{b}$)

$$\to v_i + \frac{i}{\hbar}\,\delta\varphi\,\underbrace{\hat{n}\cdot G^i}\,\vec{v}$$

how to ~~[illegible]~~ interpret this?

Instead, consider this:

$$v_i \to v_i + (\underbrace{-\varepsilon_{jik}}_{-\frac{G^j{}_{ik}}{-i\hbar}})\,\hat{n}_j\,\delta\varphi\, v_k$$

$$\to v_i - \frac{i}{\hbar}\,\delta\varphi\,\underbrace{\hat{n}_j\, G^j{}_{ik}}_{(\hat{n}\cdot\vec{G})_{ik}}\, v_k$$

: much more meaningful

$$\to v_i - \frac{i}{\hbar}\,\delta\varphi\,(G_n\vec{v})_i\,, \quad G_n := \hat{n}\cdot\vec{G}$$

$$\to \left(\delta_{ij} - \frac{i}{\hbar}\,\delta\varphi\,(G_n)_{ij}\right) v_k$$

$$\therefore\ \vec{v} \xrightarrow{\sim} e^{-\frac{i}{\hbar}\delta\varphi G_n}\,\vec{v}$$

(9)

Recall, from linear algebra that

$$e^{A} = e^{\mathcal{U}^{\dagger} D \mathcal{U}} = \mathcal{U}^{\dagger} e^{D} \mathcal{U}$$

where A is any square matrix and D is the diagonal matrix $D = \text{diag}(a_1, a_2, \dots, a_n)$ where a_i are the eigenvalues of A.

Put $A \to G_n$

$D \to J_n$

though this will work only for $\hat{n} = \hat{z}$: ($\because J_x$ or J_y is not diag.)

$$e^{-\frac{i}{\hbar}\delta\varphi G_3} = \mathcal{U}^{\dagger} e^{-\frac{i}{\hbar}\delta\varphi J_z} \mathcal{U}$$

where $\mathcal{U}$ is the very same matrix that diagonalizes G_3.

Now the phys. significance of $\mathcal{U}$: Before that, maybe I should mention about the hint at the end — photon spin.

From particle phys, we know that the "wave function" of the photon field is given by A_μ, the usual 4-potential. Apparently, this object has 4 degrees of freedom but we know that the physical degrees of freedom of the photon is 2 (to wit, $\vec{E}$ and $\vec{B}$ fields). Due to the fact that the photon field is massless,

(10)

its spin degeneracy $(2s+1 = 2(1)+1 = 3)$ reduces to 2. The physical realization of this is the 2 polarization states of light.

Recall the polarization: Since $|\vec{E}| = c|\vec{B}|$ for a 'free' light wave, the direction of $\vec{E}$ determines the polarization state. If

$$\vec{E} = |\vec{E}|\,\hat{x} \quad \text{or} \quad \vec{E} = |\vec{E}|\,\hat{y}$$

it is linearly polarized. If

$$\vec{E} = |\vec{E}|\,\frac{\hat{x} \pm i\hat{y}}{\sqrt{2}}$$

then it is RH (+) or LH (-) circular polarized.

If we treat the $\vec{v}$ vector above as the propagation vector, then we see that $\mathcal{U}$ creates a transition b/w circular and linear polarizations (just because the $\mathcal{U}$ matrix has components $(\hat{a}_+\ \hat{a}_0\ \hat{a}_-)$).

Since this part of the problem is a bit problematic, I've looked it up on the internet, But I couldn't find any satisfactory answer. The explanation above is mine and open to discussion.

(11)

3.15

(a) $$J_\pm := J_x \pm i J_y$$

$$\therefore J_x = \frac{J_+ + J_-}{2}$$

$$J_y = \frac{J_+ - J_-}{2i}$$

$$J_x^2 + J_y^2 = \frac{(J_+ + J_-)(J_+ + J_-)}{4} - \frac{(J_+ - J_-)(J_+ - J_-)}{4}$$

$$= \frac{1}{4}\Big(\cancel{J_+^2} + \cancel{J_-^2} + J_+ J_- + J_- J_+$$

$$- \cancel{J_+^2} - \cancel{J_-^2} + J_+ J_- + J_- J_+ \Big)$$

$$= \frac{1}{2}(J_+ J_- + J_- J_+)$$

$=0$

$$[J_+, J_-] = [\, J_x + iJ_y,\ J_x - iJ_y \,]$$

$=0$

$$= -i[J_x, J_y] + i[J_y, J_x]$$

$$= -2i\underbrace{[J_x, J_y]}_{i\hbar J_z}$$

$$= 2\hbar J_z$$

$$\therefore J^2 = J_x^2 + J_y^2 + J_z^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$$

$$= \frac{1}{2}\big(J_+ J_- + J_+ J_- + \underbrace{[J_-, J_+]}_{-2\hbar J_z} \big) + J_z^2$$

⑫

$$J^2 = J_+ J_- - \hbar J_z + J_z^2 \quad \text{qed}$$

(b) $J_- |jm\rangle = c_- |jm\rangle \quad \backslash \; | \; |^2$

$$\langle jm | \underbrace{J_+ J_-}_{J^2 + \hbar J_z - J_z^2} |jm\rangle = |c_-|^2 \underbrace{\langle jm|jm\rangle}_{1}$$

$$\therefore |c_-|^2 = \langle jm | J^2 - J_z^2 + \hbar J_z |jm\rangle$$

$$= \hbar^2 j(j+1) - \hbar^2 m^2 + \hbar \hbar m$$

$$= \hbar^2 \left(j(j+1) - m(m-1) \right)$$

Assume $c_- \in \mathbb{R}^+$:

$$\boxed{c_- = \hbar \sqrt{j(j+1) - m(m-1)}}$$

3.17

(a) Since the eigenfunctions of L^2 are the spherical harmonics, if we can represent x, y, and z in terms of linear combinations of Y_{lm}, then we are done. See [1]:

$$x = \sqrt{\frac{4\pi}{3}}\, y_{1-1}$$

$$y = \sqrt{\frac{4\pi}{3}}\, y_{11}$$

$$z = \sqrt{\frac{4\pi}{3}}\, y_{10}$$

$$\therefore\ \psi(\vec{x}) = (x + y + 3z)\, f(r)$$

$$= \sqrt{\frac{4\pi}{3}}\left(Y_{1-1} + Y_{11} + 3Y_{10}\right) f(r)$$

It is clear that $\boxed{l = 1}$.

(b) Since probability is a relativistic business, let's directly focus on the Y_{lm}'s:

$$\psi \propto 1\,Y_{1-1} + 1\,Y_{11} + 3\,Y_{10}$$

$$\therefore\ \left.\begin{aligned} P(m_l = -1) &= N\,1^2 = N \\ P(m_l = 1) &= N\,1^2 = N \\ P(m_l = 0) &= N\,3^2 = 9N \end{aligned}\right\}\ \sum_{m_l} P(m_l) = 1 \Rightarrow N = \frac{1}{11}$$

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(c) As a common knowledge, we know that the ~~radial~~ ~~pa~~ angular part of all wavefunctions under ~~the~~ some spherically-sym potential is the spherical harmonics.

$$\psi(\vec{x}) = \sqrt{\frac{4\pi}{3}} \left(Y_{1-1} + Y_{11} + 3Y_{10} \right) f(r)$$

Since the laplacian is a linear operator, you can collect all the spherical harmonics under a collective m:

$$Y_{1m} := \sqrt{\frac{4\pi}{3}} \left(Y_{1-1} + Y_{11} + 3Y_{10} \right)$$

so

$$\psi(\vec{x}) = f(r) Y_{1m}$$

We can do this also because the l value is unique.

Now let's "solve" the Schr. equation:

$$H\psi = E\psi$$

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \psi = E\psi$$

If you know the trick to deal w/ the laplacian, then you will realize this:

$$\frac{\vec{p}}{2m} = \frac{\vec{p_r}^2}{2m} + \frac{\vec{L}^2}{2mr^2}$$

where $p_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \right)$ and $\vec{L} \to \hbar^2 l(l+1)$ effectively.

(I recall this happens only in spherical coordinates in 3D.)

So we have

$$\left(\frac{p_r^2}{2m} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)\Bigg|_{l\to 1} f(r)\, Y_{1m}(\Omega) = E f(r)\, Y_{1m}(\Omega)$$

Since the angular derivations have been handled, we can cancel out Y_{1m}'s:

$$\left(-\frac{\hbar^2}{2m}\underbrace{\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right)\Bigg|_{l\to 1} f(r) = E f(r)$$

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) f$$

$$= \left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(f' + \frac{f}{r}\right) = f'' + \frac{f'}{r} - \frac{f}{r^2} + \frac{f'}{r} + \frac{f}{r^2}$$

$$= f'' + \frac{2f'}{r}, \quad ' := \frac{\partial}{\partial r}$$

$$-\frac{\hbar^2}{2m}\left(f'' + \frac{2f'}{r}\right) + \frac{\hbar^2}{mr^2} f + V f = E f \quad \Big\backslash \ \frac{1}{f}, \ f \neq 0 \text{ we know}$$

$$\boxed{V = E - \frac{\hbar^2}{mr^2} + \frac{\hbar^2}{2m}\left(\frac{f''}{f} + \frac{2f'}{rf}\right)}$$

[1] "Spherical harmonics," (n.d.) Retrieved from cs.dartmouth.edu/~wjarosz/publications/dissertation/appendixB.pdf

(16)

3.18

$$|\psi\rangle = |lm\rangle$$

$$L_x = \frac{L_+ + L_-}{2}, \quad L_y = \frac{L_+ - L_-}{2i}$$

$$L_\pm |lm\rangle \propto |lm\pm 1\rangle$$

$$\therefore \left.\begin{aligned} \langle L_x\rangle &= A\langle lm|lm+1\rangle + B\langle lm|lm-1\rangle = 0 \\ \langle L_y\rangle &= A'\langle lm|lm+1\rangle + B'\langle lm|lm-1\rangle = 0 \end{aligned}\right\} \boxed{\langle L_x\rangle = \langle L_y\rangle = 0}$$

qed

where A, A', B, and B' are some coefficients.

$$L_x^2 = \frac{1}{4}\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right)$$

$$L_y^2 = -\frac{1}{4}\left(L_+^2 + L_-^2 - L_+L_- - L_-L_+\right)$$

but effectively, $L_\pm^2$ terms drop, so we have

$$L_x^2 \equiv \frac{1}{4}\left(L_+L_- + \underbrace{[L_-, L_+]}_{-2\hbar L_z} + L_+L_-\right)$$

$$\equiv \frac{1}{2}\left(L_+L_- - \hbar L_z\right)$$

$$\equiv \frac{1}{2}\left(\left(L^2 + \hbar L_z - L_z^2\right) - \hbar L_z\right)$$

$$\equiv \frac{1}{2}\left(L^2 - L_z^2\right)$$

$$L_y^2 \equiv \frac{1}{4}\left(L_+L_- + L_-L_+\right)$$

$$\equiv L_x^2$$

$$\equiv \frac{1}{2}\left(L^2 - L_z^2\right)$$

$$\therefore \boxed{\langle L_x^2\rangle = \langle L_y^2\rangle = \frac{1}{2}\left(\hbar^2 l(l+1) - \hbar^2 m^2\right)}$$

qed

(17)

Semi-classical interpretation: From statistical mechanics, the average of a quantity from a single system measured "lots" of times equals that from "lots" of ~~sys~~ identically prepared systems measured once for each system. In either case, we have the following: Due to randomness (or, better, uncertainty) in that we don't know L_x and L_y values, they will cancel out (I mean, L_x will cancel among themselves, and so will L_y). This can be stated also in terms of ~~....~~ symmetry: whatever "torque" has imparted the initial angular momentum to the system, in the long-term average system preferentially picks a symmetry axis, say z (this could be x or y, as well). Since there had been no torque in the two other directions, we expect that the components of the angular mom. in those directions vanish.

Now, this perfectly explains why $\langle L_x \rangle = \langle L_y \rangle = 0$ semi-classically, but what do we do w/ $\langle L_x^2 \rangle$ and $\langle L_y^2 \rangle$? They are related to the fluctuations in the angular momentum components — that is, the RMS errors in your measurement.

(18) Recall the uncertainty is just standard deviation or the error, statistically speaking. So we have nonzero $\sqrt{\langle \Delta L_x^2 \rangle}$ and $\sqrt{\langle \Delta L_y^2 \rangle}$. They should be nonzero again from a statistical-mechanical point of view —— "everything fluctuates."

So the long story short:

- $\langle L_x \rangle = \langle L_y \rangle = 0$ $\because$ perfect cancellation of random (uncertain) components of $\vec{L}$.

- $\langle L_x^2 \rangle = \langle L_y^2 \rangle \neq 0$ $\because$ errors / deviations / fluctuations in the measurement / sys.

(due to symmetry, $\langle L_x^2 \rangle = \langle L_y^2 \rangle$ is no coincidence.)

(19)

EXTRA 6

Recall that the rotation operator in the hilbert space is given by

$$\mathcal{D}^j_n(\theta) = e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}$$

The minus sign in the exponent will be important.

(a)
$$J_3 |R,j\rangle = J_3\, e^{iJ_3\theta/\hbar} |jj\rangle$$

$$= e^{iJ_3\theta/\hbar} J_3 |jj\rangle \quad \rightarrow \text{this should be clear}$$

$$= e^{iJ_3\theta/\hbar} \hbar m |jj\rangle\Big|_{m\to j}$$

$$= \hbar j\, e^{iJ_3\theta/\hbar} |jj\rangle$$

$$= \hbar j\, |R,j\rangle$$

This tells you something quite obvious: If you rotate the system about the 3rd axis, the ~~on~~ 3rd component of the angular momentum will be conserved.

②

(b) In the usual 3D space, vectors transform as

$$\vec{v} \to R\,\vec{v}$$

and the matrices as

$$A \to R^T A R$$

under rotation. (The latter follows from this: the scalars are invariant under rotation, so

$\vec{v}\cdot A\vec{u} \to \vec{v}\cdot R^T A R\,\vec{u}$.)

Therefore in the language of quantum mechanics, we have

$$|\alpha\rangle \to \mathcal{D}(R)\,|\alpha\rangle$$

$$A \to \mathcal{D}(R)^{\dagger} A\,\mathcal{D}(R)$$

Therefore, in order to identify the Euler angles in the rotation

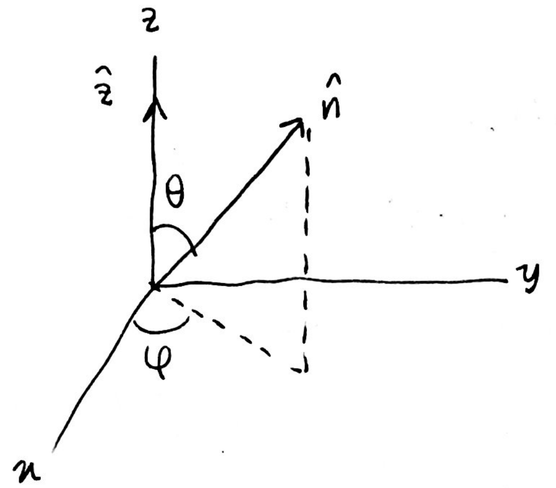
$$D^j(R)\,J_3\,D^j(R)^{-1} = \vec{J}\cdot\hat{n}$$

all you need to do is get the Euler angles in the rotation

$$R\,\hat{z} = \hat{n}$$

(Notice that $\mathcal{D}(R) = D(R)^{-1}$ in the notation of the problem. This issue will be important later.)

(21)



The standard convention for the Euler angles is

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

so let's do the following:

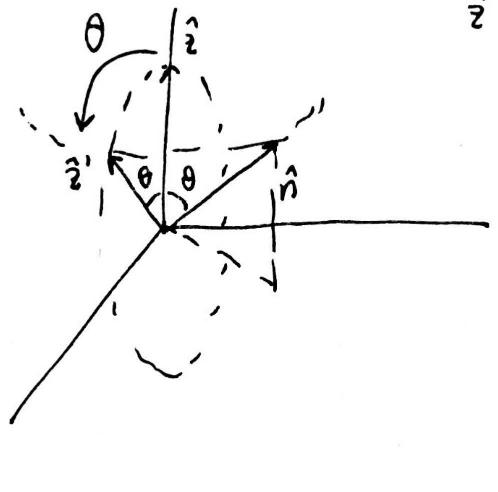
- $R_z(\gamma)\hat{z}$: $\gamma = ?$

  Since $\hat{z}$ is an eigenvector of R_z, any γ will do. Pick $\gamma = 0$.

- $R_y(\beta) R_z(0)\hat{z}$: $\beta = ?$

  ~~was~~ Clearly, $\beta = 0$:

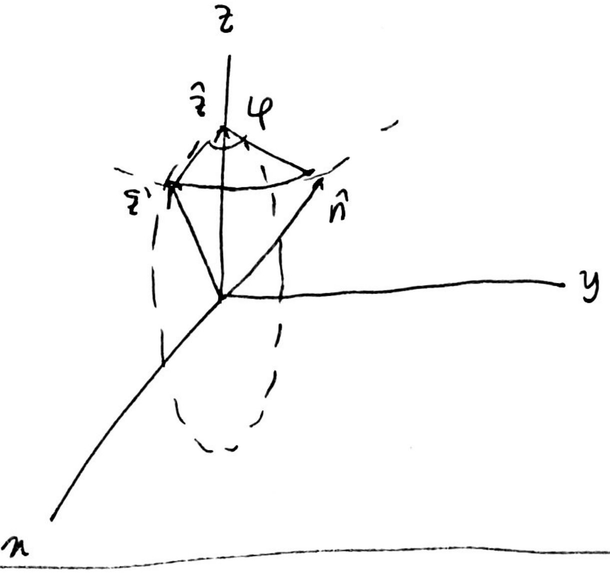
$\hat{z}' = R_y(\theta) R_z(0)\hat{z}$



· Finally,

$$R_z(\alpha)\, R_y(\theta)\, R_y(\hat{z}) = \hat{n} \quad : \quad \alpha = ?$$

clearly, $\alpha = \varphi$:



$$\therefore \boxed{\mathcal{D}(R) = \mathcal{D}_z(\varphi)\, \mathcal{D}_y(\theta)\, \mathcal{D}_z(0)}$$

or, since $\mathcal{D}(R) = D(R)^{-1}$,

$$D(R) = D_z(0)\, D_y(\theta)\, D_z(\varphi)$$

although this is a minor issue at the moment.

(23)

(i) $\vec{J} \cdot \hat{n} | R, j \rangle = D(R) J_3 D(R)^{-1} | R, j \rangle$

$$= D(R) J_3 \underbrace{D(R)^{-1} D(R)}_{1} | j j \rangle$$

$$= D(R) J_3 | j j \rangle$$

$$= D(R) \hbar j | j j \rangle$$

$$= \hbar j D(R) | j j \rangle$$

$$= \hbar j | R, j \rangle \quad \text{qed}$$